# *Essays on the Pricing and Modeling of Derivatives and Risk-taking Incentives*

**Dissertation for the Faculty of Economics, Business Administration and Information Technology at the University of Zurich**

to achieve the title of
Doctor of Philosophy
in Banking & Finance

presented by

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from Denmark

approved in January 2013 at the request of
Prof. Dr. Marc Chesney
Prof. Dr. Loriano Mancini

The Faculty of Economics, Business Administration and Information Technology of the University of Zurich hereby authorises the printing of this Doctoral Thesis, without thereby giving any opinion on the views contained therein.

Zurich, January 23, 2013

Chairman of the Doctoral Committee: Prof. Dr. Dieter Pfaff

# Declaration

I hereby declare that this dissertation is the result of my own and co-authors' work. It is being submitted in partial fulfillment for the degree of Doctor of Philosophy in Banking and Finance to the University of Zürich. It has not been submitted before for any degree or examination to any other university.

Jacob Strømberg

Zürich, Switzerland, December 13, 2012

# Acknowledgements

During my PhD studies at the Department of Banking and Finance, University of Zürich, I have had the pleasure of meeting many interesting people. I am particularly grateful to my supervisor, Prof. Marc Chesney for his invaluable support and guidance over the years. In addition, I would like to thank my co-authors Prof. Alexander F. Wagner and Prof. Markus Leippold for their support and contribution to our joint papers. I would also like to thank Eckart Jäger and Prof. Marc Paolella for their support during the first year of my PhD studies. Finally, I would like to express my gratitude to the Doctoral Program in Finance at the University of Zürich, the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK), the Swiss Finance Institute (SFI) and my supervisor Prof. Marc Chesney for financial support during my PhD studies. On a personal note, I would like to dedicate this doctoral thesis to my family.

*Jacob Strømberg*

*Zürich, Switzerland, December 13, 2012*

# Preface

The dissertation consists of three self-contained chapters in the area of quantitative finance. The first chapter is related to the modeling of LIBOR rates and the pricing of interest rate derivatives. The second chapter is related to the valuation of real options in the presence of non-diversifiable risk and the third chapter deals with the valuation of executive stock options and the embedded risk-taking incentives in managerial compensation contracts. An executive summary of each of the three chapters is provided below.

**Chapter 1:** *Time-Changed Lévy LIBOR Market Model: Pricing and Joint Estimation of the Cap Surface and Swaption Cube* (2012), joint work with Prof. Markus Leippold

One of the main challenges in term-structure modeling and derivatives pricing, is the specification of a LIBOR market model which is flexible enough to capture the joint dynamics of the implied volatilities of caps and swaptions. In the paper, we take up this challenge and design a novel and parsimonious Lévy LIBOR market model to jointly price caps and swaptions. Having access to a unique data set of implied volatilities of the cap surface[1] and the swaption cube[2] spanning the financial crisis 2007/08 we provide a comprehensive analysis of the joint pricing and estimation of our model. The model successfully matches important empirical features of the cap implied volatility surface and swaption cube. Specifically, we find a high degree of integration for longer maturity

[1] The *cap surface*, refers to the implied volatilities of caps indexed by the option maturity and the strike.

[2] The *swaption cube*, refers to implied volatilities of swaptions indexed by the expiry date of the option, the strike rate, and the tenor of the underlying swap.

caps and swaptions. We speculate, that this is due to the fact, that financial institutions and other investors primarily trade longer-dated caps and swaptions for hedging their duration exposure in the financial markets. Finally, we find a weakening link between actual and implied volatilities with mortgage backed security refinancing activity after the Fed's intervention and a strengthening link with the European sovereign debt crisis.

The originality of this work is rich. On a theoretical level, our model improves upon recent contributions to the literature (e.g., Jarrow, Li, and Zhao (2007) and Trolle and Schwartz (2009)). Furthermore, the choice of modeling co-sliding forward swap rates together with our model design allows for pricing formulae for caps and swaptions which are market-consistent with the Black model. On an empirical level, we provide a first comprehensive analysis of the joint pricing and estimation of a LIBOR market model to the cap and swaption market by incorporating information from the whole swaption cube.

One of the key challenges when designing a model to capture the joint dynamics of caps and swaptions over a period of extreme market turmoil, such as the financial crisis 2007/08, is the trade-off between model parsimony versus pricing performance. There are indications in the data, that extending the instantaneous volatility specification, e.g., to follow a two-state Markov chain would improve the pricing performance. This observation is consistent with evidence in White and Rebonato (2008) but would render the model intractable in terms of estimation. In the post-crisis period, financial market participants started to price and quote interest rate derivatives using different curves for projecting the cash flows and discounting (e.g. Mercurio (2009)). An interesting avenue for future research would be to incorporate this new practice into the model framework but it would bring along some additional model complexity.

**Chapter 2:** *Strategic Investment and Optimal Portfolio Choice under Incomplete Markets* (2012)

The aim of this paper, is to analyze how non-diversifiable risk and strategic considerations regarding technology adoption affect individuals' investment timing decision to become entrepreneurs and their optimal portfolio choice. To make progress on this issue, I develop a continuous-time real option model under incomplete markets where two risk-averse individuals strategically have to decide whether to adopt an existing technology or whether to wait for the arrival of a future technological innovation.

I show that the impact of non-diversifiable risk on the option timing decision is ambiguous and depends on the arrival intensity of the future technology. Consequently, non-diversifiable risk may accelerate or delay the optimal investment decision compared to complete markets. Moreover, higher risk-aversion leads to delayed preemption under incomplete markets whereas the impact on the follower's investment timing is ambiguous. Finally, I show that strategic considerations regarding technology adoption play a central role for the optimal portfolio choice in the presence of non-diversifiable risk. According to the model, not only current entrepreneurs being exposed to non-diversifiable income risk from managing a business should hold more conservative risky asset allocations (e.g., Heaton and Lucas (2000) for empirical evidence) it may also apply to prospective entrepreneurs.

The originality of this research, is that it challenges common predictions on investment timing from traditional real option models under complete markets. Most papers in the real option literature rely on the assumption that either the underlying real asset is directly traded or its risk profile can be spanned by trading existing financial assets. In reality, most real assets are not traded in the capital markets and their risk characteristics may at best be partially spanned by the universe of traded financial assets thus leading to incomplete markets. Recently some theoretical development has been made in the literature to extend real option models to incomplete markets. However, strategic considerations regarding investment by other individuals and arrival of future technological innovations have not yet been studied in this literature. One limitation of the model, however, relates to the utility function adopted due to analytical tractability. Future studies may explore more realistic utility specifications. Another interesting dimension to develop would be to make the innovation process endogenous within the model.

**Chapter 3:** *Managerial Incentives to Take Asset Risk* (2012), joint work with Prof. Marc Chesney and Prof. Alexander F. Wagner

The aim of this paper, is to investigate managerial incentives to take asset risk. We make progress on this issue by addressing two questions: *First*, how powerful are a typical manager's incentives to take asset risk and to increase firm value? *Second*, can asset risk-taking incentives add to our understanding of the cross-sectional variation of asset risk-taking in the financial crisis 2007/08? We answer these questions using descriptive analysis and cross-sectional regressions. The write-downs incurred by the financial institutions during the financial crisis 2007/08 serve as our dependent variable and the incentive to take asset risk acts as our prime explanatory variable in addition to a range of control variables.

We show that managerial incentives to take asset risk can be large compared to incentives to increase firm value. Moreover, we find that stock holdings can induce substantial asset risk-taking incentives, challenging the common belief regarding the central role of stock options as inducing risk-taking behavior. Finally, we find that incentives to take asset risk help explain asset risk-taking of U.S. financial institutions before the 2007/08 crisis. Using equity risk-taking incentives, instead, one may conclude that incentives do not explain risk-taking.

The originality of this research is that it challenges the traditional approach used in the literature to proxy managerial risk-taking incentives. In particular, by relying on the compound option pricing model of Geske (1979), our approach allows for a rethinking of risk-taking incentives in financial markets. For a sample of financial institutions where leverage plays a key role, we empirically document the merits of our approach. One limitation of our approach, relates to limited sample under investigation. A future analysis, may fruitfully extend our sample to include industrial companies and also adopt more general measures of risk-taking. This would altogether, facilitate more advanced econometric techniques that would allow for statements regarding potential causal effects.

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# Contents

# Chapter 1

## *Time-Changed Lévy LIBOR Market Model: Pricing and Joint Estimation of the Cap Surface and Swaption Cube*

*joint work by Markus Leippold and Jacob Strømberg*

# Time-Changed Lévy LIBOR Market Model: Pricing and Joint Estimation of the Cap Surface and Swaption Cube*

Markus Leippold† Jacob Strømberg§

Abstract

We propose a novel time-changed Lévy LIBOR (London Interbank Offered Rate) market model for jointly pricing of caps and swaptions. The time changes are split into three components. The first component allows matching the volatility term structure, the second generates stochastic volatility, and the third accommodates for stochastic skew. The parsimonious model is flexible enough to accommodate the behavior of both caps and swaptions. For the joint estimation we use a comprehensive data set spanning the financial crisis of 2007 to 2010. We find that, even during this period, neither market is as fragmented as suggested by the previous literature.

JEL-Classification: C51, E43, G13.
Keywords: LIBOR market models; Time-changed Lévy process; Caps volatilities; Swaption cube; Unscented Kalman filter.

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## I. Introduction

We introduce a novel time-changed Lévy LIBOR (London Interbank Offered Rate) market model (LMM) that is analytically tractable and parsimonious. Yet, it is flexible enough to jointly and consistently price caps and swaptions in an efficient way. For the period spanning the financial crisis of 2007 to 2010, we perform a comprehensive empirical analysis of our model. We make use of a unique data set of implied volatilities of the cap surface and the entire swaption cube. The swaption cube refers to the implied volatilities of swaptions indexed by the expiry date of the option, the strike rate, and the tenor of the underlying swap. Previous endeavors on the joint pricing of caps and swaptions restrict their swaption data to at-the-money (ATM) quotes. However, we incorporate all information contained in non-ATM volatilities for estimating the model.[1] To our best knowledge, this is the first paper that develops and estimates a model that jointly prices the whole swaption cube and the caps implied volatility surface.

To design our theoretical model, we start with a preliminary data analysis of the cap volatility surface and the swaption cube. We then introduce three distinct model devices to match the stylized features of the data. The first component is a Brownian motion combined with a parametric function, which allows us to capture the volatility term structure. The second component is a time-changed Brownian motion that generates stochastic volatility. Correlating the time change with the changes in the LIBOR gives us additional flexibility to match implied volatility skews. The third component is a time-changed jump process with asymmetric upward and downward jumps. This component allows us to accommodate variations in volatility skews over time. Hence, these three components not only match different characteristics of the implied volatilities, but also allow for a parsimonious parameterization. The parsimony is crucial for both efficient pricing and model estimation.

The modeling approach we take belongs to the general class of time-changed Lévy processes.

[1] Data on the swaption cube have not been available to researchers until recently. To our knowledge, Trolle and Schwartz (2013) are the first to systematically analyze data from the swaption cube.

Our motivation to use this class lies in its generality.[2] On the one hand, Lévy processes can generate almost any return innovation distribution and they can account for potential discontinuities in the LIBOR dynamics.[3] On the other hand, applying time changes randomizes the innovation distribution of LIBORs over time. Imposing suitable time changes allows us to match volatility term structure effects, to generate stochastic volatility, and to accommodate for stochastic skew. While the literature on Lévy term structure models has been exclusively concerned with presenting a theoretically consistent no-arbitrage framework for pricing derivatives, no guidance has been given on the empirical performance of these models. We fill this gap and bring our time-changed Lévy model to the data.

We estimate our model using a maximum likelihood method joint with the unscented Kalman filter. Analyzing our parameter estimates, we find a strong negative correlation between changes in volatility and changes in LIBORs. This negative correlation could be driven by hedging activities in the market for mortgage-backed securities (MBS). Previous literature argues that, as interest rates drop and borrowers prepay their mortgages, the increasing hedging activity of government-sponsored institutions, such as the Federal National Mortgage Association (Fannie Mae) and the Federal Home Loan Mortgage Corporation (Freddie Mac), could lead to an increase in interest rate volatilities, particularly for longer maturities.[4]

To examine whether the markets of caps and swaptions are integrated during the recent financial crisis, we carefully analyze the characteristics of the pricing errors. Our empirical analysis suggests that caps and swaptions markets are well integrated during the financial crisis, especially when

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[2] Lévy models, but without time changes have been adopted for modeling interest rates within the class of LMMs by, e.g., Eberlein and Raible (1999), Eberlein and Ozkan (2005), Eberlein and Kluge (2006), and Eberlein and Liinev (2007).

[3] Several papers have empirically shown the need for incorporating jumps in interest rates, e.g., Babbs and Webber (1997), Das (2002), El-Jahel, Lindberg, and Perraudin (1997), and Johannes (2004).

[4] Duarte (2008) shows that the inclusion of prepayment speed as an additional factor in the LIBOR model of Longstaff, Santa-Clara, and Schwartz (2001) significantly reduces the pricing error of ATM swaptions. These results are further corroborated by Li and Zhao (2009). However, hedging demand from the MBS market could have decreased when on January 5, 2009 the Fed began purchasing fixed rate mortgage-backed securities guaranteed by Fannie Mae, Freddie Mac, and Ginnie Mae (Government National Mortgage Association).

we look at contracts with intermediate and longer maturities. In contrast, several papers find segmentation between the two markets.[5] Such findings are more likely driven by the rigidity of the model used, because our parsimonious and yet flexible model can accommodate the variations of the two markets very well.

We proceed as follows. In Section II, we describe the data and present a preliminary data analysis on the implied volatilities of caps and the swaption cube, which guide our model design. We introduce the time-changed Lévy LIBOR model in Section III and show how to construct the family of forward LIBOR and swap rates. In Section IV, we adopt the fast Fourier transform (FFT) technique to price interest rate derivatives within our Lévy framework. In Section V, we present our estimation strategy based on the unscented Kalman filter. In Section VI, we present the results of our estimation exercise, and Section VII concludes.

## II. Data analysis

We obtain the cap and floor implied volatility mid-quotes on the US dollar three-month forward LIBOR from Bloomberg (ICAP), covering a wide range of strikes and maturities. The implied volatility quotes are available for ten fixed maturities that include every year from one to ten years. At each date and maturity, we have seven fixed strike levels, including 1.5%, 2%, 3%, 4%, 5%, 6%, and 7% and also one floating strike level at the money. We obtain the swaptions implied volatility data from BGC Partners.[6] The ATM swaption quotes were collected for option maturities equal to three and six months and one, two, three, four, five, seven, and ten years. These option maturities are combined with swap terms of one, two, three, four, five, seven, and ten years. We end up with a total of 63 ATM swaptions implied volatility quotes per observation day. For out-of-the-money and in-the-money volatilities, we collect option maturities equal to three months, one year, five years,

[5]See, e.g., Jagannathan, Kaplin, and Sun (2003), Longstaff, Santa-Clara, and Schwartz (2001), Brigo and Mercurio (2002), Driessen, Klaassen, and Melenberg (2003), and Fan, Gupta, and Ritchken (2003).

[6]BGC Partners, Inc. is a leading global intermediary for wholesale financial markets, specializing in brokering a broad range of financial products, including fixed income and interest rates. For a further description of the swaption cube, including liquidity issues, refer to Trolle and Schwartz (2013).

and ten years. These option maturities are combined with swap terms of two, five, and ten years, spanning strikes of $\pm\{25, 50, 100, 200\}$ basis points away from the ATM swaption quotes.

Our data set spans the period from August 8, 2007 to August 11, 2010, covering three years of data. This includes the recent financial crisis and thus serves as an interesting period over which to test our model. For the estimation period, we consider weekly data sampled on Wednesdays to avoid weekday effects. The summary statistics of the LIBORs and swap rates are well documented. Therefore, we focus on the behavior of the cap and floor and swaptions implied volatility quotes along three important dimensions: moneyness, maturity (of the option as well as the swap), and time.

### II.1. Time-series dynamics of the implied volatilities of ATM caps and swaptions

Fig. 1 shows the evolution of the implied volatilities of ATM caps and swaptions spanning our data sample period. Significant time variation exists in the implied volatilities for both caps and swaptions. From levels of around 20% before the crisis, the implied volatilities increased dramatically, reaching levels above 100% for the shorter-dated option maturity contracts. Because implied volatilities are quoted under the Black (1976) model, which assumes log-normally distributed forward LIBORs and swap rates, this increase might not only result from markets being more volatile, but also from interest rates going down.

The implied volatilities in the two markets exhibit similar co-movements over time. This behavior is natural because caps and swaptions essentially share the same underlying quantity. The forward swap rates driving swaption prices can be viewed as a weighted sum of forward LIBORs, which drive cap prices. The correlation of the one-year caps and swaptions ATM implied volatilities is 95% over our sample period. Furthermore, the average correlation between the ATM implied volatilities of caps and swaptions across all option and swap maturities averages 84%. However, the correlation decreases with increasing maturity of the underlying swap. Hence, swaps with longer maturities carry some term structure information that is not contained in the LIBOR underlying the cap

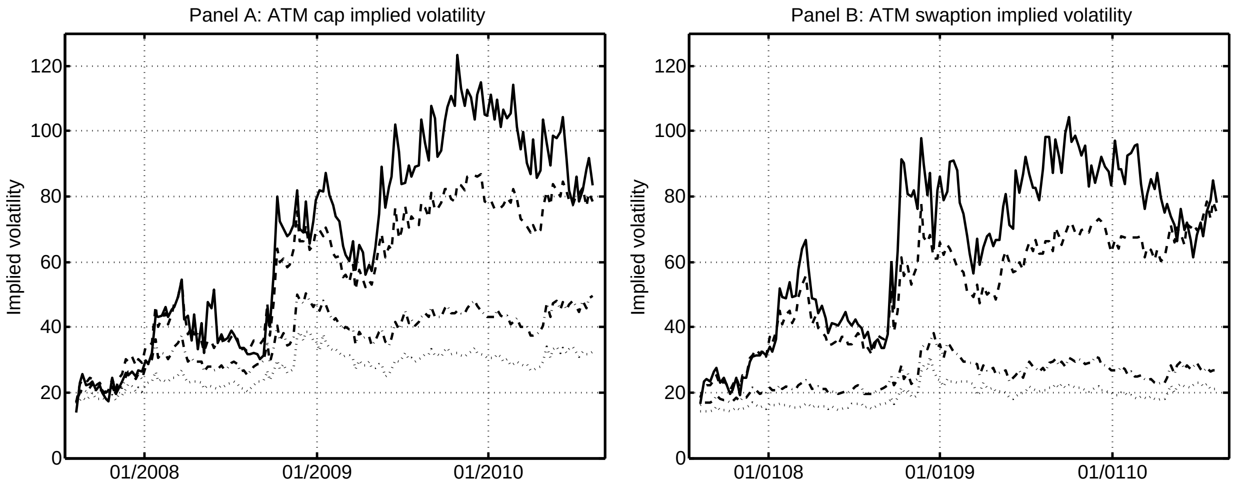


**Figure 1.** Time variation in caps and swaptions at-the-money (ATM) implied volatilities. The figure shows the time-variation in ATM cap (Panel A) and swaption (Panel B) implied volatilities in percentage points. For caps, we use option maturities of one (solid line), two (dashed line), five (dash-dotted line), and ten years (dotted line). For swaptions, we use option maturities of three months, one year, five years, and ten years, for a one year swap term. The data are weekly (Wednesday), spanning our entire data sample from August 8, 2007 to August 11, 2010; in total, 158 weeks.

prices. Nevertheless, the strong co-movement at shorter swap maturities suggests these two markets should be analyzed jointly, as they could be significantly integrated.

## II.2. Volatility term structure across option maturities

When we analyze swaption data, we have two different maturity dimensions: the maturity of the option contract and the maturity or tenor of the underlying swap. We discuss only the option maturity dimension and keep the swap tenor fixed.

In Fig. 2, we plot the evolution of the volatility term structure for ATM caps (Panel A) and for ATM swaptions on a one-year swap (Panel B). During our sample period, the term structure of volatilities is usually monotonically decreasing for both caps and swaptions. Also, the term structure for caps tends to be higher than for swaptions. However, both curves exhibit from time to time a

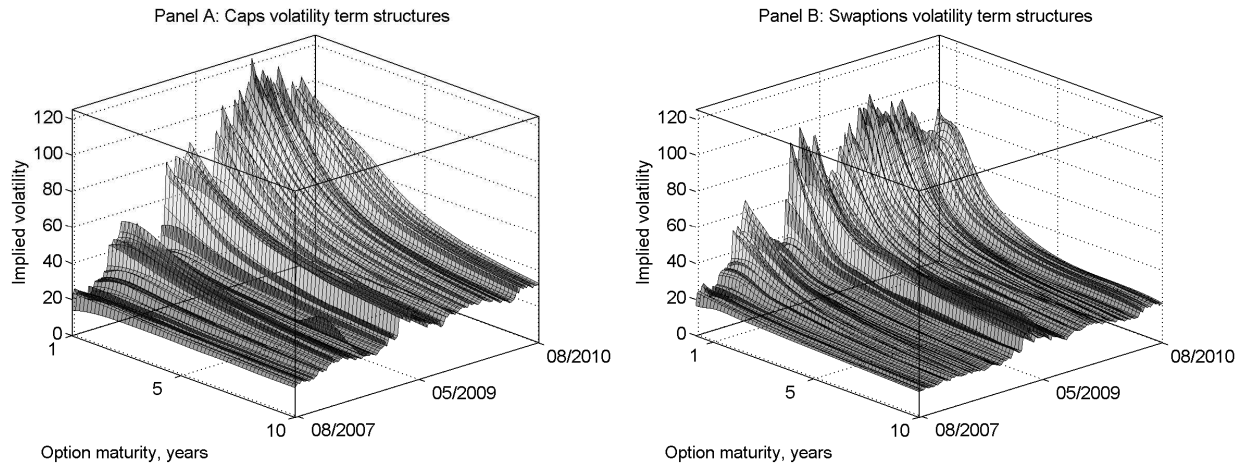


**Figure 2.** Evolution of implied volatility term structures. Panel A shows the evolution of the at-the-money (ATM) volatility term structure for caps implied volatilities in percentage points across the one- to ten-year option maturities. Panel B shows the evolution of the ATM volatility term structure for swaptions implied volatilities across the three-month to ten-year option maturities for the one-year swap tenor.

hump-shaped term structure. Hence, as indicated in Fig. 2, term structure shapes for both markets could simultaneously share similar patterns through time. If this is the case, then a joint estimation using data from both markets could facilitate identification of the model.

## II.3. Cap and swaptions implied volatility smile/skew across moneyness

For our graphical analysis of implied volatilities across moneyness, we use a quadratic fit to the quoted implied volatilities, for which we standardize the moneyness $d$ by

$$d \equiv \frac{\ln K/S(t,T)}{ATMV(t,T)\sqrt{T-t}}, \tag{1}$$

where $K$ is the strike rate level, $S(t,T)$ denotes the swap rate of the corresponding maturity (spot for caps and forward starting for swaptions), $ATMV(t,T)$ is the ATM implied volatility quote, and $T-t$ denotes the option's time-to-maturity. Because the cap is a portfolio of caplets, the moneyness for different caplets differs due to differences in both maturities and underlying forward rates. Hence, for caps, the above definition of moneyness represents an aggregate approximate measure. We perform a quadratic fit as follows:

$$IV = \widehat{a} + \widehat{b}\, d + \widehat{c}\, d^2 + \epsilon. \tag{2}$$

The slope estimate $\widehat{b}$ measures the implied skewness, and the curvature estimate $\widehat{c}$ captures the implied kurtosis in the distribution of the underlying rate. The estimate $\widehat{a}$ captures the average level of the implied volatilities and $\epsilon$ denotes the residual between the actual implied volatility quotes and the quadratic fit.

Fig. 3 plots the caps and swaptions implied volatility smiles and skews at an arbitrary day together with their average shapes. On November 28, 2007 (Fig. 3, Panels A and B), the implied volatilities exhibit significant skewness at each maturity. November 28, 2007 is just at the onset of the financial crisis. At the time, significant uncertainty existed about the current and future levels of the forward LIBOR, which drives the amount of interbank lending. The relatively high caps implied volatilities for low strikes indicate that financial institutions were demanding caps with low strikes, presumably as an insurance or hedge against future hikes in the LIBOR.

A similar pattern, although to a lesser extent, emerges in Panel B for swaption prices. Here, we plot the implied volatilities for swaptions on the five-year swap. These volatilities are quoted at a lower level than the corresponding volatilities for caps, which corresponds to the intuition that the LIBOR underlying the cap should be more volatile than the swap rate, as it has a shorter maturity (three months). Nevertheless, it seems that, on that particular day, both markets shared the same qualitative characteristics in terms of the volatility shape. This observation is confirmed when we look at the average shapes of the implied volatilities across moneyness in Panels C and D of Fig. 3. They are all negatively skewed for both markets. The cap volatilities are higher on average than the swaption volatilities. Furthermore, they tend to have a more pronounced skew and, at least for the caps volatilities with one-year maturity, the implied volatility has a slight smile pattern.

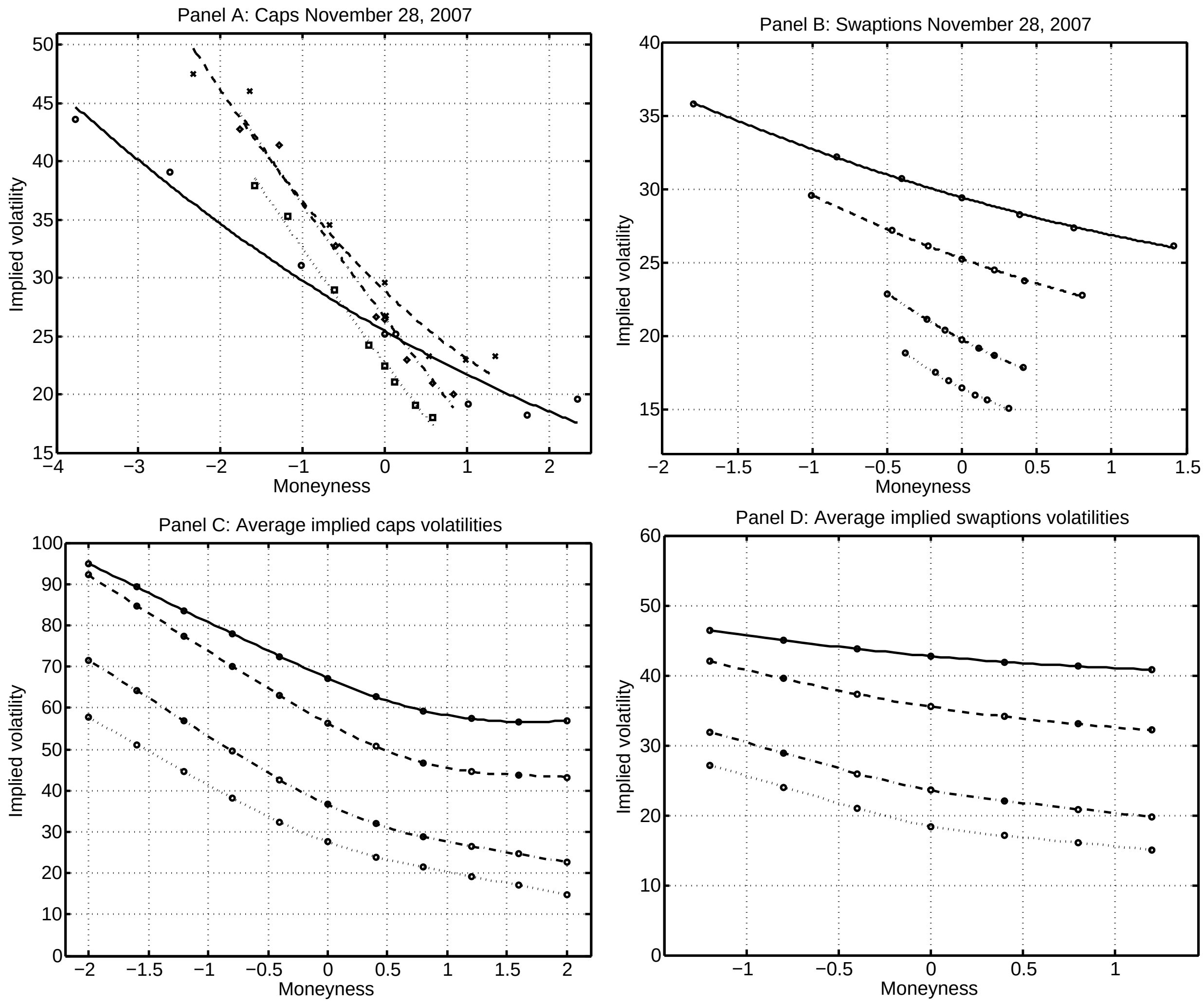


**Figure 3.** Cap and swaptions implied volatility smiles and skews across moneyness. In Panel A, we plot caps implied volatility smiles in percentage points on November 28, 2007. The four lines in Panel A correspond to four maturities: one year (solid line), two years (dashed line), five years (dash-dotted line), and ten years (dotted line). Circles are data points; lines are quadratic fits. In Panel B, we plot swaptions implied volatility smiles on November 28, 2007, for a five-year swap tenor. The four lines in each panel represent four option maturities: three months (solid line), one year (dashed line), five years (dash-dotted line), and ten years (dotted line). In Panels C and D, we plot the corresponding implied volatility smiles for caps and swaptions when averaged across our whole data sample.

### II.4. Dynamic properties of the caps and swaptions implied skew and curvature

For our model design, we could gain further insights into the properties of caps and swaptions dynamics by analyzing the slope and curvature of the implied volatilities over time. As in Subsection 2.3, we summarize each caps and swaptions implied volatility smile by three quantities: its level $\widehat{a}$, its slope $\widehat{b}$, and its curvature $\widehat{c}$ from the quadratic fit in Eq. (2) normalized by the ATM volatility level.

In Fig. 4, we plot the time series behavior of the slope and curvature of caps and swaptions implied volatility for an option maturity of five years. For the underlying swap, we consider a tenor of two years. Panel A shows a cap volatility skew that is systematically higher than the swaption skew. Nevertheless, they move in the same direction. Hence, we can conclude that the forward rate as well as the swap rate exhibit negative skewness and that their skewness is highly correlated.

By inspection of Panel B, we find that the curvature behaves slightly differently. At the beginning of the sample period, the curvature of the caps is lower than that of the swaptions. In particular, the time series plot indicates that the forward rates did not have much excess kurtosis until January 2008. In the subsequent period, the excess kurtosis increased steadily to levels similar to the excess kurtosis of the two-year swap rate. Furthermore, the variation in the caps' curvature, and hence in the forward rate's excess kurtosis, is much more pronounced. Nevertheless, the curvature for the implied volatilities of caps and swaptions seems to be correlated as well. Repeating the same calculations across different option maturities, swap tenors, and moneyness, we observe that the correlations between skew and curvature are substantial, especially for intermediate and long option maturities.

Hence, by incorporating information from the whole swaption cube, we do not necessarily have to introduce a large number of additional factors to fully capture the moneyness and the two maturity dimensions. Instead, given the high co-movement, especially across the maturity dimensions, the swaption cube might improve the model identification when it comes to estimation.

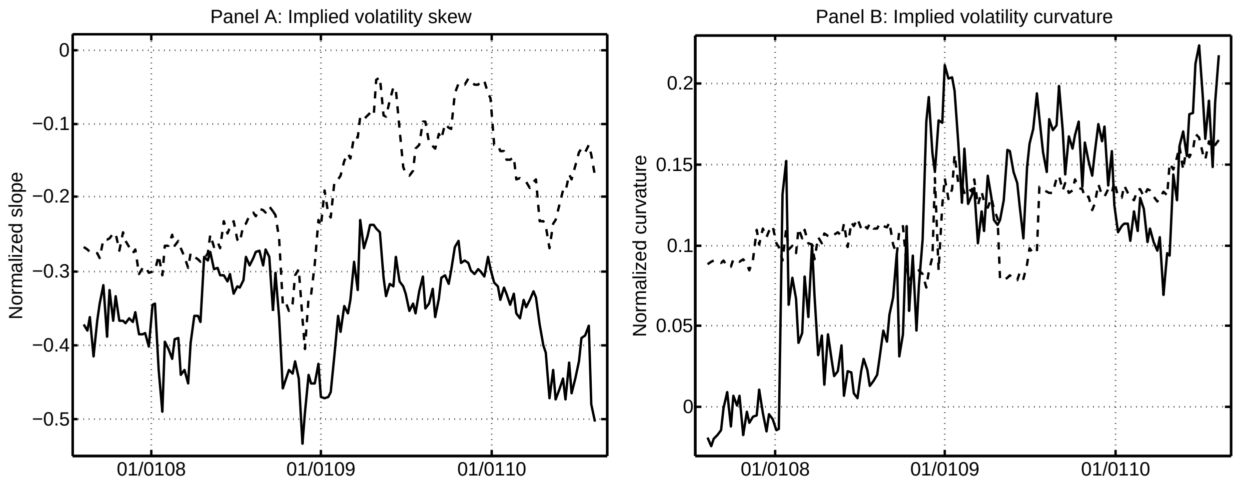


**Figure 4.** Time variation in caps and swaptions implied volatility skew and curvature. The figure plots the time series of the normalized slope and curvature of the caps and swaptions implied volatility smile. In Panel A, we plot the implied volatility skew of a five-year cap (solid line) and for a five-year swaption to enter into a two-year swap (dashed line). In Panel B, we plot, for the same contracts, the time series of the corresponding curvatures. The caps and swaptions skews and curvatures have been fitted using weekly (Wednesday) data, spanning the period from August 8, 2007 to August 11, 2010.

## II.5. Principal component analysis

Now that we have analyzed the data across different maturities and levels of moneyness, the next step is to analyze how many factors are needed to jointly describe the caps and swaptions data in the cross-sectional and time series dimension. We do so by using a principal component analysis (PCA). As in Heidari and Wu (2003), we first perform a PCA of the LIBOR and swap rates to identify the common factors driving the yield curve.[7] Once we have extracted the common factors from the interest rate market, we regress the caps and swaptions implied volatilities on the yield curve factors. We use the regression residuals to perform another PCA to identify the common factors driving the implied volatilities. Finally, we regress the caps and swaptions implied volatilities on

[7]However, we do not introduce their modification to explicitly account for the sharp difference in liquidities between the interest rate market and the swaptions market.

both the yield curve and the common volatility factors.

**Table I.** Principal component analysis (PCA).

| | Number of term structure and volatility factors $(m, n)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (1,1) | (1,2) | (2,1) | (2,2) | (2,3) | (3,1) | (3,2) | (3,3) |
| Caps | 89.7 | 94.9 | 93.3 | 97.4 | 98.0 | 95.0 | 97.7 | 98.3 |
| ATM swaptions | 89.6 | 92.0 | 91.2 | 93.8 | 94.6 | 94.1 | 95.9 | 96.6 |
| Non-ATM swaptions | 86.0 | 89.2 | 87.7 | 92.0 | 93.6 | 91.6 | 94.6 | 96.1 |
| Total | 88.5 | 92.1 | 90.8 | 94.4 | 95.4 | 93.6 | 96.1 | 97.0 |

$R^2$ values (in percentages) are from regressing the implied volatilities of caps and swaptions on the yield curve factors and the volatility factors. We obtain the yield curve factors from a PCA on LIBORs (London Interbank Offered Rates) and swap rates. The volatility factors are obtained from a PCA on the residuals of regressing the implied volatilities on the yield curve factors. The table shows the $R^2$ values (in percentage) for different factor combinations $(m, n)$, where $m$ is the number of yield curve factors and $n$ is the number of volatility factors. The numbers are averaged for caps, at-the-money (ATM) swaptions, and non-ATM swaptions. We base our analysis on weekly data sampled on Wednesdays, spanning the period August 8, 2007 to August 11, 2010.

Table I shows the resulting $R^2$ values. Because we have in total 161 time series of implied volatilities, we take averages across caps, ATM swaptions, and non-ATM swaptions. We analyze different factor combinations $(m, n)$, where $m$ is the number of yield curve factors and $n$ is the number of volatility factors. On average, a model with one term structure and one volatility factor already explains 88.5% of the variance in caps and swaptions implied volatilities. However, especially for non-ATM swaptions, models that do not include more than three factors cannot explain more than 90% of the variation. When we add factors, the explained variation for all implied volatilities increases well above 93%. The $(3, 3)$-model with three term structure and three volatility factors explains 97%. This result is very close to the findings in Heidari and Wu (2003). However, they analyze only ATM swaptions data.

We also see from Table I that the $(3, 2)$-model with three term structures and two volatility factors does a reasonably good job in explaining the variation across the different markets. The

**Table II.** Explained variation for swaptions across option maturities and swap tenors for the (3,2)-model.

*Panel A: $R^2$ for ATM swaptions*

| | *Option maturity* | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *Swap tenor* | 3 m | 6 m | 1 y | 2 y | 3 y | 4 y | 5 y | 7 y | 10 y |
| 1 y | 94.4 | 97.0 | 97.4 | 95.4 | 95.9 | 95.3 | 95.1 | 94.3 | 91.9 |
| 2 y | 94.9 | 96.7 | 96.3 | 95.7 | 96.0 | 95.7 | 95.7 | 95.4 | 92.1 |
| 3 y | 95.6 | 96.1 | 96.9 | 96.8 | 96.5 | 96.5 | 96.1 | 96.0 | 91.4 |
| 4 y | 95.4 | 96.8 | 97.8 | 97.4 | 97.2 | 97.0 | 96.6 | 96.0 | 90.8 |
| 5 y | 95.1 | 97.4 | 98.1 | 97.7 | 97.6 | 97.3 | 96.9 | 95.7 | 90.3 |
| 7 y | 96.5 | 97.9 | 98.4 | 97.8 | 97.6 | 97.2 | 96.7 | 95.1 | 90.8 |
| 10 y | 95.8 | 97.0 | 97.7 | 97.4 | 97.1 | 96.6 | 95.9 | 94.2 | 91.3 |

*Panel B: $R^2$ for non-ATM swaptions*

| | *Option maturity* | | | | *Swap tenor* | | |
|---|---|---|---|---|---|---|---|
| *Moneyness* | 3 m | 1 y | 5 y | 10 y | 2 y | 5 y | 10 y |
| 0.90 | 93.1 | 97.3 | 96.0 | 90.0 | 92.6 | 94.8 | 94.9 |
| 0.95 | 93.3 | 97.5 | 96.2 | 91.0 | 93.1 | 95.3 | 95.2 |
| 1.05 | 93.5 | 97.6 | 96.3 | 91.7 | 93.4 | 95.6 | 95.3 |
| 1.10 | 93.5 | 97.5 | 96.2 | 92.2 | 93.6 | 95.8 | 95.3 |

Panel A shows the $R^2$ values from regressing the (3,2)-model with three term structure and two volatility factors on at-the-money (ATM ) swaptions volatilities. Panel B shows the corresponding $R^2$ values for non-ATM volatilities. We base our analysis on weekly data sampled on Wednesdays, spanning the period August 8, 2007 to August 11, 2010.

total variation explained is above 96% and for non-ATM swaptions close to 95%. Therefore, for the sake of parsimony, we content ourselves with designing a model that is based on two instead of three volatility factors. To further validate such a factor structure, we can break down the explained variation across the different caps and swaptions implied volatilities for the $(3,2)$-model. Because this model already explains 97.7% of the variation in caps implied volatilities, in Table II, we present the results only for the swaptions volatilities.

Table II shows the $R^2$ values for swaptions across different option maturities and swap tenors (Panel A) and across different levels of moneyness (Panel B). For the ATM swaptions in Panel A, we find the comforting result that all entries are above 90%. The performance of the $(3, 2)$-model struggles most for the ten-year option maturity with some values only slightly above 90%. We also note that the $R^2$ values are more stable across swap tenors than across option maturities, which indicates that the volatility factors could act differently mainly along the option maturity dimension, while their impact along the swap tenor dimension is relatively flat. Again, this observation is in line with Heidari and Wu (2003). For the non-ATM swaptions in Panel B, we find a similar pattern. For a given level of moneyness and if compared across option maturities, the variation of the $R^2$ values is larger than the variation of $R^2$ values across swap tenors. However, even for non-ATM swaptions, the $R^2$ values never fall below 90% for the $(3, 2)$-model.

## III. The specification of the Lévy LIBOR market model

Guided by the stylized features of the caps' implied volatility surface and the swaption cube, we next introduce our term structure model. Instead of starting from instantaneous forward rates or zero-coupon bond prices, we begin directly with the specification of the forward LIBORs and model them as Lévy processes. To describe the tenor structure of the forward LIBORs, we consider a fixed set of increasing and equidistant maturities $T_0, T_1, \ldots, T_n$ with $\delta \equiv T_{j+1} - T_j$ for all $j$. The maturity $T_n$ denotes the terminal maturity and the tenor $\delta$ is typically three months. We assume the existence of an initial strictly positive and decreasing term structure of zero-coupon bonds, $P(0, T)$, for $T \in (0, T_n]$. We denote by $L(0, T_j) \equiv L(0; T_j, T_{j+1})$ the forward LIBOR contracted at date $t = 0$ for the period $[T_j, T_{j+1}]$ defined by

$$L(0, T_j) \equiv \frac{1}{T_{j+1} - T_j} \left( \frac{P(0, T_j)}{P(0, T_{j+1})} - 1 \right), \quad j = 0, \ldots, n-1. \tag{3}$$

## III.1. Model design

We consider a complete filtered probability space $\{\Omega, \mathcal{F}_{T_n}, (\mathcal{F}_t)_{0\leq t\leq T_n}, \mathbb{P}\}$ with the augmented filtration $(\mathcal{F}_t)_{0\leq t\leq T_n}$ satisfying the usual conditions. As it is convenient to price interest rate derivatives under the so-called $T$-forward measure $\mathbb{Q}_n \sim \mathbb{P}$, we specify the dynamics of the forward LIBOR directly under $\mathbb{Q}_n$. Under this measure, $L(t, T_{n-1})$ is a martingale on the interval $[0, T_{n-1}]$ and we assume that it has the representation

$$L(t, T_{n-1}) \quad = \quad L(0, T_{n-1}) \exp\left(\int_0^t b^{\mathbb{Q}_n}(s, T_{n-1}, T_n)ds + \int_0^t dX_s\right), \tag{4}$$

where

$$\begin{aligned} X_s \quad &= \quad \int_0^t \lambda(s, T_{n-1})dB_s^{\mathbb{Q}_n} + \int_0^t \sqrt{V_s^W}dW_s^{\mathbb{Q}_n} + \int_0^t \int_{-\infty}^0 x\left(\mu^-(ds, dx) - \pi_{J^-}^{\mathbb{Q}_n}(x)dx\nu_s^J ds\right) \\ &+ \quad \int_0^t \int_0^\infty x\left(\mu^+(ds, dx) - \pi_{J^+}^{\mathbb{Q}_n}(x)dx\nu_s^J ds\right) \end{aligned} \tag{5}$$

is a nonhomogeneous Lévy process. The term $b^{\mathbb{Q}_n}(t, T_{n-1}, T_n)$ in Eq. (4) denotes a deterministic drift, which we need to specify in such a way that the forward LIBOR $L(t, T_{n-1})$ becomes a $\mathbb{Q}_n$-martingale. We decompose the random shocks in the above forward LIBOR dynamics into different types. First, we introduce continuous shocks by two standard Brownian motions $B_t^{\mathbb{Q}_n}$ and $W_t^{\mathbb{Q}_n}$, for which we assume $dB_t^{\mathbb{Q}_n}dW_t^{\mathbb{Q}_n} = 0$. Second, we introduce discontinuous shocks $J_t$, for which we allow a separate specification of positive jumps $J_t^+$ and negative jumps $J_t^-$, i.e., $J_t = J_t^+ + J_t^-$. By $\mu^+(dt, dx)$ and $\mu^-(dt, dx)$, we denote the counting measures for upward and downward jumps, respectively. By $\pi_{J^+}^{\mathbb{Q}_n}(x)$ and $\pi_{J^-}^{\mathbb{Q}_n}(x)$, we denote the corresponding Lévy densities, which characterize the jump structure under $\mathbb{Q}_n$. The arrival rate of upward jumps of size $x$ at time $t$ is governed by $\pi_{J^+}^{\mathbb{Q}_n}(x)\nu_t^J$. Hence the $\mathbb{Q}_n$-compensator for upward jumps becomes $\pi_{J^+}^{\mathbb{Q}_n}(x)dx\nu_t^J dt$. The compensator for downward jumps has the same form, but with $\pi_{J^+}^{\mathbb{Q}_n}(x)$ replaced by $\pi_{J^-}^{\mathbb{Q}_n}(x)$.[8]

[8] A similar specification of the jump components has recently been proposed by Carr and Wu (2011) for modeling equity index options.

### III.1.1. Time changes

The specification of the LIBOR in Eqs. (4) and (5) is a nonhomogenous Lévy process under $\mathbb{Q}_n$, defined by its time-dependent local characteristic triplet. This time dependency can also be interpreted through the following time changes:

$$\mathcal{T}_t^B = \int_0^t \lambda(s, T_{n-1})^2 ds, \qquad \mathcal{T}_t^W = \int_0^t V_s^W ds, \qquad \mathcal{T}_t^J = \int_0^t \nu_s^J ds. \tag{6}$$

We can specify these three components separately. First, we specify the time changes $\mathcal{T}_t^W$ and $\mathcal{T}_t^J$ through their activity rates under $\mathbb{Q}_n$:

$$dV_t^W = \kappa_W(\theta_W - V_t^W)dt + \sigma_W \sqrt{V_t^W} d\widetilde{W}_t^{\mathbb{Q}_n}, \qquad dW_t^{\mathbb{Q}_n} d\widetilde{W}_t^{\mathbb{Q}_n} = \rho dt \tag{7}$$

and

$$d\nu_t^J = \kappa_J(\theta_J - \nu_t^J)dt + \sigma_J \sqrt{\nu_t^J} dZ_t^{\mathbb{Q}_n}, \tag{8}$$

where $\widetilde{W}_t^{\mathbb{Q}_n}$ and $Z_t^{\mathbb{Q}_n}$ are $\mathbb{Q}_n$-Brownian motions. We assume that the instantaneous correlation between the time changes of the jump and the continuous component is zero. The specification in Eqs. (7) and (8) dictates that stochastic volatility enters our model via two different sources: (1) the time change of the Brownian motion $W_t^{\mathbb{Q}_n}$ and (2) the stochastic activity rate of jumps $\nu_t^J$. Furthermore, to account for a potential leverage effect, we allow the instantaneous correlation between the activity rate $V_t^W$ and the Brownian motion $W_t^{\mathbb{Q}_n}$ to be nonzero.

In addition to the time changes $\mathcal{T}_t^W$ and $\mathcal{T}_t^J$, we introduce a purely deterministic time change $\mathcal{T}_t^B$ governed by the parametric functional form of $\lambda(t, T_{n-1})$. By choosing a functional form such as

$$\lambda(t, T_j) = (\beta_1 + \beta_2(T_j - t)) \exp(-\beta_3(T_j - t)) + \beta_4, \tag{9}$$

we can ensure not only sufficient flexibility for the volatility function to match potentially hump-shaped patterns (see, e.g., Rebonato, McKay, and White, 2009), but also analytical tractability.[9]

### III.1.2. Jump process

For the jump specification, we borrow the variance-gamma jump process from Carr and Wu (2007). They propose a simple yet flexible specification for the Lévy density that successfully reconciles the properties of currency option skews. We split the Lévy density under the terminal forward measure $\mathbb{Q}_n$ into a right-skewed jump component and a left-skewed jump component:

$$\pi_J^{\mathbb{Q}_n}(x) = \begin{cases} \pi_{J^+}^{\mathbb{Q}_n}(x) = \lambda e^{-\frac{x}{\nu_+}} x^{-1}, & x > 0 \\ \pi_{J^-}^{\mathbb{Q}_n}(x) = \lambda e^{-\frac{|x|}{\nu_-}} |x|^{-1}, & x < 0 \end{cases}, \tag{10}$$

where $\lambda, \nu_+, \nu_- > 0$. Hence, we let the jump arrival rate decrease monotonically with increasing jump size. The parameters $\nu_+$ and $\nu_-$ control the scale of the positive and negative jumps. Applying different scales allows us to capture any asymmetric discontinuous movements in the jump arrival rate of the forward LIBOR. Because the characteristic exponent of the jump process in Eq. (10) is available in closed form, our LIBOR model remains analytically tractable.

Before we continue with constructing the family of forward LIBORs and swap rates, note that our additive structure not only ensures the analytical tractability of the characteristic exponent of the underlying Lévy process, which is crucial for deriving our pricing formulae, but the explicit mapping of the various components, to capture specific properties of caps and swaptions implied volatilities, also facilitates the economic interpretation of the model. First, we have seen in the preliminary data analysis of Section II that we need a time-varying stochastic volatility in the underlying forward LIBOR (see Fig. 1). We let the process $V_t$ take care of this. Second, as Fig. 2 illustrates, we also need a sufficiently general and flexible volatility function to capture various shapes of the volatility term structure, such as monotonically decreasing and hump-shaped forms. Therefore,

[9] Note that $\lim_{t\to T}\lambda(t,T) = \beta_1 + \beta_4$ and $\lim_{T\to\infty}\lambda(t,T) = \beta_4$. Furthermore, the maximum of the volatility curve occurs at a $T^*$, where $T^* = 1/\beta_3 - \beta_1/\beta_2$.

we have chosen the functional form of $\lambda(t,T)$ as in Eq. (9). Third, Fig. 3 provides evidence for implied volatility smiles and skews in both the caps and swaptions market, underscoring the importance of allowing for a nonzero correlation between innovations to the forward rates and its underlying stochastic volatility (see, e.g., Casassus, Collin-Dufresne, and Goldstein (2005)). Fourth, the findings in Fig. 4 call for a model that can match both the persistent fat-tail behavior and the strong time variation in the skewness (specifically for short option maturities) of the forward-neutral distribution of the forward log-LIBOR. The fat-tail behavior could be captured by including a jump component into the underlying forward rate dynamics. As Carr and Wu (2007) remark, standard jump-diffusion models have difficulty in generating strong time variation in the risk-neutral skewness. By randomizing the clock for the Lévy jump component as we do in our model specification, we can introduce stochastic skewness into the underlying distribution of the forward log-LIBOR. Finally, the principal component analysis (see Subsection II.5) indicates that our model specification should be rich enough to match the data across the dimensions of the option maturity, the swap tenor, and the moneyness in both the caps and swaptions market. Hence, while keeping the model parsimonious with three term structure factors and two volatility factors, each model device has its own individual role in matching the stylized facts of caps and swaptions volatilities.

### III.1.3. Market price of risk

To capture the observed time series dynamics under the historical measure (represented by $\mathbb{P}$), we need to specify the different market prices of risk related to each stochastic component in our model. However, because the forward LIBOR and swap rate processes under the measure $\mathbb{P}$ are not relevant for option pricing, we leave the market price of return risk unspecified and focus on the activity rate processes. For simplicity, we assume constant market prices of risk for the activity rates. In particular, we define the measure change from the terminal forward LIBOR measure $\mathbb{Q}_n$ to $\mathbb{P}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}_n}_{|\mathcal{F}_t} \quad = \quad \mathcal{E}\left(-\gamma^V \widetilde{W}_{\mathcal{T}_t^W}\right)\mathcal{E}\left(-\gamma^\nu Z_{\mathcal{T}_t^J}\right). \tag{11}$$

### III.2. Constructing the family of forward LIBOR rates

In the representation of the forward LIBOR $L(t, T_{n-1})$ in Eqs. (4) and (5), the deterministic term $b^{\mathbb{Q}_n}(t, T_{n-1}, T_n)$ corresponds to a convexity-adjustment related to the Brownian motions and to the jump components, respectively. For any $j = 0, 1, \ldots, n-1$, $L(t, T_j)$ has to be a martingale under the $T_{j+1}$-forward measure $\mathbb{Q}_{j+1}$, which we can ensure by a corresponding adjustment of the term $b^{\mathbb{Q}_{j+1}}(t, T_j, T_{j+1})$. Proposition 1 shows how this martingale property can be enforced.[10]

**Proposition 1.** *The forward LIBOR $L(t, T_j)$ is a martingale under the $T_{j+1}$-forward measure for $j = 0, \ldots, n-1$ if the following drift condition is satisfied:*

$$b^{\mathbb{Q}_{j+1}}(t, T_j, T_{j+1}) \quad = \quad -\frac{1}{2}\lambda^2(t, T_j) - \frac{1}{2}V_t^W - \int_{-\infty}^{\infty} (e^x - 1 - x)\, \pi_J^{\mathbb{Q}_{j+1}}(dx)\nu_t^J. \tag{12}$$

Proposition 1 allows us to express the martingale dynamics of the forward LIBOR $L(t, T_j)$ under the $T_{j+1}$-forward measure by

$$\begin{aligned} \frac{dL(t, T_j)}{L(t, T_j)} \quad &= \quad \lambda(t, T_j) dB_t^{\mathbb{Q}_{j+1}} + \sqrt{V_t^W} dW_t^{\mathbb{Q}_{j+1}} + \int_{-\infty}^{0} (e^x - 1) \left( \mu^-(dt, dx) - \pi_{J^-}^{\mathbb{Q}_{j+1}}(x) dx \nu_t^J dt \right) \\ &+ \quad \int_0^{\infty} (e^x - 1) \left( \mu^+(dt, dx) - \pi_{J^+}^{\mathbb{Q}_{j+1}}(x) dx \nu_t^J dt \right), \end{aligned} \tag{13}$$

subject to the activity rate processes in Eqs. (6) to (9) and the jump specification given in Eq. (10) under the appropriate forward measure.

For a recursive backward construction of the family of forward LIBORs $L(t, T_j)$ under their corresponding forward measures $\mathbb{Q}_{j+1}$, we have to switch between different forward measures. For that purpose, we need to know how the components of the nonhomogenous Lévy process $X_t$ must be adjusted for different forward measures. Proposition 2 shows how to do so.[11]

**Proposition 2.** *For each $j = 2, \ldots, n$, the forward measure defining the martingale dynamics of*

[10] We relegate all proofs to Appendix A.

[11] For regularity conditions, we refer to Eberlein and Ozkan (2005).

*the family of forward LIBORs is related to the terminal forward measure* $\mathbb{Q}_n$ *by*

$$
\begin{aligned}
dB_t^{\mathbb{Q}_{j-1}} &= dB_t^{\mathbb{Q}_n} - \sum_{k=1}^{n+1-j} \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \lambda(t, T_{n-k}) dt, && (14) \\
dW_t^{\mathbb{Q}_{j-1}} &= dW_t^{\mathbb{Q}_n} - \sum_{k=1}^{n+1-j} \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \sqrt{V_t^W} dt, && (15)
\end{aligned}
$$

*and*

$$
\pi_J^{\mathbb{Q}_{j-1}} \nu_t^J = \prod_{k=1}^{n+1-j} \left(1 + \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \left(e^x - 1\right)\right) \pi_J^{\mathbb{Q}_n} \nu_t^J, \tag{16}
$$

*where* $V_t^W$, $\nu_t^J$, $\lambda(t, T_j)$, *and* $\pi_J^{\mathbb{Q}_n}$ *are defined in Eqs.* (7) *to* (10)*.*

Next, we need to find the change of measure of the activity rate dynamics under the forward measures $\mathbb{Q}_{j+1}$ governing the stochastic volatility $V_t^W$ that affects the Brownian motion $W_t$ due to the nonzero correlation assumption. The result is given in Proposition 3.

**Proposition 3.** *The forward changes of measure related to the Brownian motion* $\widetilde{W}^{\mathbb{Q}_{j-1}}$ *of the stochastic activity rate in Eq.* (7) *obey*

$$
d\widetilde{W}_t^{\mathbb{Q}_{j-1}} = d\widetilde{W}_t^{\mathbb{Q}_n} - \sum_{k=1}^{n+1-j} \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \sqrt{V_t^W} \rho dt, \quad j = 2, \ldots, n. \tag{17}
$$

Propositions 2 and 3 provide us with the results necessary for the construction of the family of forward LIBORs $L(t, T_j)_{t \in [0, T_j]}$ driven by the nonhomogenous Lévy process in Eq. (5) under their corresponding forward measures $\mathbb{Q}_{j+1}$ for $j = 0, \ldots, n-1$.

### III.3. Constructing the family of forward swap rates

For the construction of the forward swap rates, we assume a set of equally spaced reset dates $\{T_0, T_1, \ldots, T_n\}$ with interval length $\delta$. For a swap contract starting at date $T_j$, $j = 1, \ldots, n - N$, the first settlement date is at time $T_{j+1}$ and the maturity date of the swap is $T_{N+j}$. We fix the length $N$ of the swap, but we let the starting date, and thus also the maturity, vary. We let $R_j^N(t) \equiv R(t; T_j, T_{j+N})$ denote the forward swap rate contracted at time $t$ for the swap arrangement over the period $[T_j, T_{j+N}]$. In terms of zero bond prices, the swap rate equals

$$R(t; T_j, T_{j+N}) = \frac{P(t, T_j) - P(t, T_{j+N})}{\delta \sum_{k=j+1}^{j+N} P(t, T_k)} = \frac{P(t, T_j) - P(t, T_{j+N})}{S_t(T_j, T_{j+N})}, \quad j = 0, \ldots, n-N. \tag{18}$$

The term $S_t(T_j, T_{j+N}) = \sum_{k=j+1}^{j+N} \delta P(t, T_k)$ is often referred to as the present value of a basis point or the sliding level process.[12] In what follows, we make use of an approximate representation of the swap rate. The representation, which is given in Proposition 4, helps us retain analytical tractability.[13]

**Proposition 4.** *The terminal co-sliding forward swap rate $R_{n-N}^N(t)$ given by*

$$R_{n-N}^N(t) \quad \approx \quad R_{n-N}^N(0) \exp\left( \int_0^t \sum_{k=n-N}^{n-1} \tilde{\omega}_k(0) dX_s + \mathit{drift} \right) \tag{19}$$

*is a $\mathbb{Q}_{n-N+1}^N$-martingale.*[14] *The weights*

$$\tilde{\omega}_k(0) = \frac{\omega_k(0) L(0, T_k)}{R_j^N(0)}, \tag{20}$$

*with $\omega_k(0) = \frac{P(0, T_{k+1})}{\sum_{k=j+1}^{j+N} P(0, T_k)}$, are obtained by freezing the coefficients to their initial values. Under*

[12] See, e.g., Bjork (2004).

[13] We use an approximation based on freezing the coefficients together with an approximation of exponential functions. The technique of freezing the coefficients is well established and has been tested for its quality by, e.g., Brigo and Mercurio (2006, Chapter 8). For details, refer to Appendix A.

[14] The *drift* in Eq. (19) has to be adjusted in such a way that $R_{n-N}^N(t)$ is a $\mathbb{Q}_{n-N+1}^N$-martingale. Because it has no impact on the pricing, we leave it unspecified.

*the terminal co-sliding forward swap measure* $\mathbb{Q}^N_{n-N+1}$*, we have*

$$\begin{aligned} X_s &= \int_0^t \lambda(s,T_k)dB_s^{\mathbb{Q}^N_{n-N+1}} + \int_0^t \sqrt{V_s^W} dW_s^{\mathbb{Q}^N_{n-N+1}} + \int_0^t \int_{-\infty}^0 x\left(\mu^-(ds,dx) - \pi_{J^-}^{\mathbb{Q}^N_{n-N+1}}(x)dx\nu_s^J ds\right) \\ &+ \int_0^t \int_0^\infty x\left(\mu^+(ds,dx) - \pi_{J^+}^{\mathbb{Q}^N_{n-N+1}}(x)dx\nu_s^J ds\right). \end{aligned} \quad (21)$$

*The jump intensity under* $\mathbb{Q}^N_{n-N+1}$ *becomes*

$$\pi_J^{\mathbb{Q}^N_{n-N+1}}(x)\nu_s^J = e^{\varphi_2 x}\pi_J^{\mathbb{Q}_n}(x)\nu_s^J \quad (22)$$

*for* $J = \{J^-, J^+\}$*, and the activity rate dynamics are*

$$dV_t^W = \left(\kappa_W\theta_W - \tilde{\kappa}^{\mathbb{Q}^N_{n-N+1}}V_t^W\right)dt + \sigma_W\sqrt{V_t^W}d\widetilde{W}_t^{\mathbb{Q}^N_{n-N+1}} \quad (23)$$

$$d\nu_t^J = \kappa_J(\theta_J - \nu_t^J)dt + \sigma_J\sqrt{\nu_t^J}dZ_t^{\mathbb{Q}^N_{n-N+1}}, \quad (24)$$

*with* $\tilde{\kappa}^{\mathbb{Q}^N_{n-N+1}} = \kappa_W - \varphi_2\sigma_W\rho$ *for a given* $N$*, and where* $\varphi_2$ *is defined in Lemma A.2 of Appendix A.*

The above specification of the forward swap rates with fixed $N$ is also referred to as a co-sliding forward swap rate model.[15] This specification facilitates the estimation and calibration of our model to actual market quotes, and, most important, it is consistent with our previous construction of forward LIBORs. The forward LIBORs $L(t;T_j,T_{j+1})$ coincides with the one-period forward swap rate over the interval $[T_j, T_{j+1}]$. Hence, we can view the LIBOR market model as a subclass of the co-sliding forward swap rate model with $N = 1$ corresponding to the three-month tenor. For a given length of the swap tenor $N$ we can, with minimal effort, adjust all measure change results from Subsection III.2 and apply them to the co-sliding forward swap rate model. In contrast, if we were to specify a swap market model based on, e.g., the co-terminal swap rate, the theoretical results and

[15] See, e.g., Musiela and Rutkowski (2005), Chapter 13.5.

the numerical implementation would become substantially more involved and cumbersome.[16]

From the specification of the co-sliding forward swap rate in Proposition 4, we can start the backward construction of the entire family of co-sliding forward swap rates. But because the co-sliding forward swap rate in Eq. (19) is an exponential of nonhomogenous Lévy processes under the appropriate co-sliding forward swap rate measure, the calculations are essentially similar to those for the LIBOR model. Therefore, we omit the details of the construction of the family of co-sliding forward swap rates. The explicit calculations can be obtained upon request.

## IV. Pricing interest rate derivatives

For pricing interest rate derivatives, we adopt the FFT technique introduced for stock options by Carr and Madan (1999).

### IV.1. Pricing caps

A cap contract is a sum of a number of basic contracts known as caplets. Each caplet can be viewed as a call option on the underlying forward LIBOR such that the time $t$ value of the cap with maturity $T_n$ and strike $K$ can be represented by

$$Cap_t(K,T_n) = \sum_{j=0}^{n-1} P(t,T_{j+1})\mathbb{E}_t^{\mathbb{Q}_{j+1}}\left(\delta(L(T_j,T_j)-K)^+\right) \quad (25)$$

under the forward measure $\mathbb{Q}_{j+1}$. We can write the time $t$ price of a caplet on the forward LIBOR $L(T_j,T_j)$ with strike $K$ and maturity $T_j$ as

$$C_t(K,T_j) = \delta P(t,T_{j+1})\mathbb{E}_t^{\mathbb{Q}_{j+1}}\left[(L(T_j,T_j)-K)^+\right] = \delta P(t,T_{j+1})\mathbb{E}_t^{\mathbb{Q}_{j+1}}\left[(e^{Y_{T_j}}-e^k)^+\right], \quad (26)$$

[16] Eberlein and Liinev (2007), derive measure change formulae for a Lévy swap rate model based on co-terminal forward swap rates. Galluccio, Ly, Scaillet, and Huang (2007) show that the LIBOR market model is the only admissible model of a co-sliding type.

where $k = \ln K$ and

$$Y_{T_j} \;=\; \ln L(t,T_j) + \int_t^{T_j} b^{\mathbb{Q}_{j+1}}(s,T_j,T_{j+1})ds + \int_t^{T_j} dX_s. \tag{27}$$

The drift $b^{\mathbb{Q}_{j+1}}(t,T_j,T_{j+1})$ is given in Proposition 1. To calculate cap prices, we make use of the complex valued measure change as developed in Carr and Wu (2004). Furthermore, for the characteristic exponent of the convexity-adjusted jump component of the Lévy process $\psi_J^{T_{j+1}}(u)$ under the measure $\mathbb{Q}_{j+1}$, we apply the widely used freezing coefficients approach. The result is given in Proposition 5.

**Proposition 5.** *The time $t$ price of the caplet given in Eq. (26) can be calculated by*

$$C_t(k,T_j) \;=\; \frac{e^{-z_i k}}{\pi}\delta P(t,T_{j+1})\int_0^\infty e^{-iz_r k}\frac{\phi_{Y_{T_j}}(z_r - iz_i - i)}{(iz_r+z_i)(iz_r+z_i+1)}dz_r. \tag{28}$$

*The characteristic function $\phi_{Y_{T_j}}(\cdot)$ of the nonhomogenous Lévy process $Y_{T_j}$ is given by*

$$\phi_{Y_{T_j}}(u) \;=\; \mathbb{E}_t^{\mathbb{Q}_{j+1}}\left(\exp\left(iuY_{T_j}\right)\right) \tag{29}$$

$$= L(t,T_j)^{iu}\exp\left[-\frac{1}{2}\left(iu+u^2\right)\int_t^{T_j}\lambda(s,T_j)^2ds - a_W(\tau) - b_W(\tau)V_t - a_J(\tau) - b_J(\tau)\nu_t^J\right], \tag{30}$$

*where $\tau = T_j - t$ and the coefficients $a_W(\tau)$, $b_W(\tau)$, $a_J(\tau)$, and $b_J(\tau)$ are given by*

$$a_i(\tau) \;=\; \frac{\kappa_i\theta_i}{\sigma_i^2}\left[2\ln\left(1-\frac{\gamma_i-\hat{\kappa}_i}{2\gamma_i}\left(1-e^{-\gamma_i\tau}\right)\right)+(\gamma_i-\hat{\kappa}_i)\,\tau\right], \tag{31}$$

*and*

$$b_i(\tau) \;=\; \frac{2\psi_i(u)\left(1-e^{-\gamma_i\tau}\right)}{2\gamma_i-(\gamma_i-\hat{\kappa}_i)\left(1-e^{-\gamma_i\tau}\right)}, \qquad \gamma_i = \sqrt{\hat{\kappa}_i^2+2\sigma_i^2\psi_i(u)}. \tag{32}$$

*For the index $i \in \{W, J\}$ and*

$$\psi_i(u) = \begin{cases} \psi_W(u) = \frac{1}{2}(iu + u^2) & \text{if } i = W \\ \psi_J^{\mathbb{Q}_{j+1}}(u) & \text{if } i = J \end{cases}, \qquad \hat{\kappa}_i = \begin{cases} \kappa_j^{\mathbb{M}} & \text{if } i = W \\ \kappa_J & \text{if } i = J \end{cases}, \tag{33}$$

*with* $\kappa_j^{\mathbb{M}} = \kappa_W - \sum_{k=1}^{n-j-1} \frac{\delta L(0,T_{n-k})}{1+\delta L(0,T_{n-k})}\sigma_W \rho - iu\sigma_W \rho$, *and*

$$\psi_J^{\mathbb{Q}_{j+1}}(u) \approx \int_{\mathbb{R}_0} \left(1 - e^{iux}\right) \prod_{k=1}^{n-1-j} \left(1 + \frac{\delta L(0, T_{n-k})}{\delta L(0, T_{n-k}) + 1} \left(e^x - 1\right)\right) \pi_J^{\mathbb{Q}_n} dx. \tag{34}$$

The characteristic exponent $\psi_J^{\mathbb{Q}_{j+1}}(u)$ can be recursively calculated in closed form starting from the terminal measure $\mathbb{Q}_n$. The term $\int_t^{T_j} \lambda(s, T_j)^2 ds$ can also be calculated in closed form. Therefore, our model specification allows closed form solutions for pricing caps and floors (up to a single integration), which enables us to use FFT methods efficiently.

### IV.2. Pricing swaptions

A payer swaption (PS) is a call option that allows us to enter into an interest rate swap agreement at some future time. The term "payer" refers to the fixed leg of the contract, such that, when we enter into a swap agreement, we receive the floating leg and pay the fixed leg. We consider a payer swaption on the forward swap rate $R_j^N(T_j)$ with strike $K$. The value under the co-sliding forward swap measure $\mathbb{Q}_{j+1}^N$ at time $t$ is

$$\begin{aligned} PS_t(T_j, T_{j+N}, K) &= \sum_{k=j+1}^{j+N} \delta P(t, T_k) \mathbb{E}_t^{\mathbb{Q}_{j+1}^N} \left[ (R_j^N(T_j) - K)^+ \right] && (35) \\ &= S_t(T_j, T_{j+N}) \mathbb{E}_t^{\mathbb{Q}_{j+1}^N} \left[ (R_j^N(T_j) - K)^+ \right], && (36) \end{aligned}$$

for $j = 0, 1, \ldots, n - N$. Under the measure $\mathbb{Q}_{j+1}^N$, the forward swap rate $R_j^N(t)$ is a martingale. Because we work under the approximate swap rate derived in Proposition 4, we use the pricing formula

$$PS_t(T_j, T_{j+N}, K) \approx S_t(T_j, T_{j+N}) \mathbb{E}_t^{\mathbb{Q}_{j+1}^N} \left[ \left( e^{\widetilde{Y}_{T_j}} - e^k \right)^+ \right], \tag{37}$$

where $k = \ln K$ and

$$\widetilde{Y}_{T_j} = \ln R_j^N(t) + \int_0^{T_j} \sum_{k=j}^{j+N-1} \tilde{\omega}_k(0) dX_s + drift, \tag{38}$$

and where the weights $\tilde{\omega}_k(0)$ are given in Eq. (20). Because the precise specification of the drift in $\widetilde{Y}_{T_j}$ is not relevant for the option pricing formula, we simply write it as *drift*.[17] The price of a payer swaption is given in Proposition 6.

**Proposition 6.** *The time $t$ price of a payer swaption with maturity $T_j$ on the swap rate $R_j^N(T_j)$ with strike $K$ is approximately given by*

$$PS_t(k, T_j, T_{j+N}) \approx \frac{e^{-z_i k}}{\pi} S_t(T_j, T_{j+N}) \int_0^\infty e^{-iz_r k} \frac{\phi_{\widetilde{Y}_{T_j}}(z_r - iz_i - i)}{(iz_r + z_i)(iz_r + z_i + 1)} dz_r. \tag{39}$$

*The characteristic function $\phi_{\widetilde{Y}_{T_j}}(\cdot)$ of $\widetilde{Y}_{T_j}$ is given by*

$$\begin{aligned} \phi_{\widetilde{Y}_{T_j}}(u) &= \mathbb{E}_t^{\mathbb{Q}_{j+1}^N} \left( \exp\left( iu\widetilde{Y}_{T_j} \right) \right) && (40) \\ &= \exp\left( -\frac{1}{2} \left( iu + u^2 \right) \int_t^{T_j} \left( \sum_{k=j}^{j+N-1} \tilde{\omega}_k(0) \lambda(s, T_k) \right)^2 ds \right) \\ &\times R_j^N(t)^{iu} \exp\left( -a_W(\tau) - b_W(\tau) V_t - a_J(\tau) - b_J(\tau) \nu_t^J \right), && (41) \end{aligned}$$

[17] Again, for the derivation of the swaption price in the Proposition 6, we make use of the freezing coefficients technique.

*where* $\tau = T_j - t$. *The coefficients* $a_W(\tau)$, $b_W(\tau)$, $a_J(\tau)$, *and* $b_J(\tau)$ *are given by*

$$a_i(\tau) \quad = \quad \frac{\kappa_i \theta_i}{\sigma_i^2} \left[ 2 \ln \left( 1 - \frac{\gamma_i - \hat{\kappa}_i}{2\gamma_i} \left( 1 - e^{-\gamma_i \tau} \right) \right) + (\gamma_i - \hat{\kappa}_i)\, \tau \right] \tag{42}$$

*and*

$$b_i(\tau) \quad = \quad \frac{2\psi_i(u) \left( 1 - e^{-\gamma_i \tau} \right)}{2\gamma_i - (\gamma_i - \hat{\kappa}_i) \left( 1 - e^{-\gamma_i \tau} \right)}, \qquad \gamma_i = \sqrt{\hat{\kappa}_i^2 + 2\sigma_i^2 \psi_i(u)}, \tag{43}$$

*and, for the index* $i \in \{W, J\}$,

$$\psi_i(u) = \begin{cases} \psi_W(u) = \frac{1}{2}(iu + u^2) & \text{if } i = W \\ \psi_J^{\mathbb{Q}_{j+1}^N}(u) & \text{if } i = J \end{cases} \qquad\qquad \hat{\kappa}_i = \begin{cases} \kappa_j^{\mathbb{M}} & \text{if } i = W \\ \kappa_J & \text{if } i = J \end{cases}, \tag{44}$$

*with* $\kappa_j^{\mathbb{M}} = \kappa_W - \varphi_2 \sigma_W \rho - \sum_{k=1}^{n-N-j} \frac{\delta R_{n-N+1-k}^N(0)}{1+\delta R_{n-N+1-k}^N(0)} \sigma_W \rho - iu\sigma_W \rho$. *The characteristic exponents of the convexity-adjusted jump component are given by the approximation*

$$\psi_J^{\mathbb{Q}_{j+1}^N}(u) \quad \approx \quad \int_{\mathbb{R}_0} \left( 1 - e^{iux} \right) \prod_{k=1}^{n-N-j} \left( 1 + \frac{\delta R_{n-N+1-k}^N(0)}{1 + \delta R_{n-N+1-k}^N(0)} \left( e^x - 1 \right) \right) \pi_J^{\mathbb{Q}_{n-N+1}^N} dx, \tag{45}$$

*where*

$$\pi_J^{\mathbb{Q}_{n-N+1}^N}(x) = \begin{cases} \lambda e^{-\frac{x}{\bar{\nu}_+}} x^{-1}, & x > 0 \\ \lambda e^{-\frac{|x|}{\bar{\nu}_-}} |x|^{-1}, & x < 0 \end{cases}, \tag{46}$$

*with* $\bar{\nu}_+ = \frac{\nu_+}{1-\varphi_2 \nu_+}$, $\bar{\nu}_- = \frac{\nu_-}{1+\varphi_2 \nu_-}$, *and* $\varphi_2$ *given in Lemma A.2 in Appendix A.*

Again, not only can the term $\int_t^{T_j} \left( \sum_{k=j}^{j+N-1} \tilde{\omega}_k(0) \lambda(s, T_k) \right)^2 ds$ be calculated in closed form, but so can $\psi_J^{\mathbb{Q}_{j+1}^N}(u)$. Furthermore, a special case of the above valuation formula is obtained for the case $N = 1$. With $N = 1$, the swap rate is equal to the three-month LIBOR. Because for $N = 1$ the term $\varphi_2$ vanishes, the pricing result in Proposition 6 corresponds to the one in Proposition 5 for caplets. Hence, in concluding this section, we can say that our flexible yet parsimonious time-changed Lévy model allows analytical pricing for caps as well as for swaptions. Furthermore, the resulting formulae are consistent with each other.

## V. Estimation

Implied volatilities for caps and swaptions are quoted under the Black (1976) model, which assumes log-normally distributed forward LIBORs and swap rates. To convert the implied volatility quotes into the option prices used in the estimation, we need the zero-coupon bond prices (see also Appendix B for details). We obtain from Bloomberg the US dollar forward LIBORs at maturities of one, two, three, six, nine, and 12 months and US dollar swap rates with various maturities. Relying on the Nelson and Siegel (1987) parametric form, we bootstraped the forward three-month forward LIBOR curve.

For the estimation of our model, we follow other studies on pricing interest rate derivatives (e.g., Jarrow, Li, and Zhao, 2007) and define the moneyness of a contract as the ratio between the strike and the ATM strike of the particular contract (cap or swaption). For caps we consider moneyness spanning from 0.80 to 1.20 with intervals of 0.10. For swaptions, due to lower liquidity in the cross section, we consider a range of moneyness from 0.90 to 1.10 with intervals of 0.05. In total, we end up with 161 interest rate derivative quotes (50 cap and floor and 111 swaption quotes) spanning three years of data (158 weeks), which yields a total of $25'438$ quotes to be matched in the joint estimation.

The model uses two state variables, namely, the activity rates $V_t$ and $\nu_t$, to capture the variation of the implied volatility surface in the caps and swaptions market over time. To estimate the model parameters, we cast the model into state space form by treating the state variables as hidden states. The implied volatility quotes from the caps and swaptions market serve as observations with errors. We employ a nonlinear filter, the unscented Kalman filter, to extract the levels of the states at each date in our sample.[18] The model parameters are estimated by maximizing the likelihood defined on the forecasting errors of option prices.

For the state space formulation, we treat the two activity rates as unobservable state variables. Hence, their dynamics constitute the state propagation equations, which we formulate in discrete time as

$$[V_{t+1}, \nu_{t+1}]' = f(V_t, \nu_t; \Theta) + \sqrt{\Sigma}\varepsilon_{t+1}, \tag{47}$$

where

$$f(V_t, \nu_t; \Theta) = \begin{pmatrix} \kappa_W \theta_W \Delta t + (1 - \kappa_W^{\mathbb{P}} \Delta t) V_t^W \\ \kappa_J \theta_J \Delta t + (1 - \kappa_J^{\mathbb{P}} \Delta t) \nu_t^J \end{pmatrix}, \qquad \Sigma_t = \begin{pmatrix} \sigma_W^2 V_t^W & 0 \\ 0 & \sigma_J^2 \nu_t^J \end{pmatrix} \Delta t, \tag{48}$$

with $\kappa_W^{\mathbb{P}} = \kappa_W - \sigma_W \gamma^V$ and $\kappa_J^{\mathbb{P}} = \kappa_J - \sigma_J \gamma^\nu$. We denote by $\Delta t = 7/365$ the weekly frequency of the data we apply in the estimation. The term $\varepsilon_{t+1}$ denotes an independent and identically distributed *(i.i.d.)* bivariate standard normal innovation. For the measurement equation, we write

$$y_t = h(V_t, \nu_t; \Theta) + \sqrt{\Omega} e_t, \tag{49}$$

where the vector $y_t$ contains the observed market prices of caps, floors, and swaptions at time $t$ scaled by their respective vega, i.e., their sensitivity to volatility changes. The function $h(\cdot)$ denotes the model-implied option prices as a function of our parameter set $\Theta$ and the state vector $[V_t, \nu_t]'$. To obtain cap, floor and swaption prices from the implied volatility quotes, we invert them by using the Black (1976) model. Appendix B presents the details on how we compute the option prices from the market data. We assume the pricing errors $e_t$, are independent and normally distributed with zero mean and constant diagonal covariance matrix $\Omega$.

[18]For a general treatment of unscented Kalman filters, see Wan and Van Der Merwe (2000) and Leippold and Wu (2007) for an application to term structure modeling.

To estimate the model parameters, we define the quasi log-likelihood value for each week's observation of the option prices assuming that the forecasting errors are normally distributed:

$$l_{t+1}(\Theta) = -\frac{1}{2}\ln\left|\overline{A}_t\right| - \frac{1}{2}\left(\left(y_{t+1} - \overline{y}_{t+1}\right)^\top \left(\overline{A}_{t+1}\right)^{-1}\left(y_{t+1} - \overline{y}_{t+1}\right)\right), \tag{50}$$

where $\overline{y}$ denotes the model-implied option prices and $\overline{A}$ denotes the covariance on the model-implied option prices. We choose the model parameters to maximize the log-likelihood of the data series, which is a summation of the weekly log-likelihood values

$$\hat{\Theta} \equiv \arg\max_{\Theta} \mathcal{L}(\Theta, \{y_t\}_{t=1}^N), \quad \text{with} \quad \mathcal{L}(\Theta, \{y_t\}_{t=1}^N) = \sum_{t=0}^{N-1} l_{t+1}(\Theta), \tag{51}$$

where $N = 158$ is the number of weeks in our sample, $\Theta$ denotes the parameter set to be estimated, and $\hat{\Theta}$ denotes the optimal parameters. Our full model specification has 15 parameters,

$$\Theta = \left\{\beta_1, \beta_2, \beta_3, \beta_4, \kappa_W, \theta_W, \sigma_W, \kappa_J, \theta_J, \sigma_J, \rho, \nu_+, \nu_-, \gamma^V, \gamma^\nu\right\}, \tag{52}$$

which govern the dynamics of the underlying forward LIBORs and swap rates and two state variables $\{V_t, \nu_t\}$ to price the time series and cross-sectional behavior of 161 interest rate options each week corresponding to a total of $25'438$ contracts.

Despite its richness in economic structures and flexibility in allowing separate sources of volatility variation, the model is very parsimonious in terms of the number of free parameters. In particular, its parsimony and the large number of option prices available allow us to identify the model parameters with strong statistical significance.

## VI. Results

In the following, we discuss our estimation results for the theoretical model developed in the previous sections.

### VI.1. Parameter estimates

Table III gives the parameter estimates together with their standard errors and the log-likelihood. Consistent with our discussion in Subsection III.1, we find that a large and significantly negative correlation parameter $\rho$ is needed to fit the implied volatility skew in both the caps and swaptions market. In the estimation we get $\rho = -0.844$. If we take the sample average of the activity rate $V_t$, our estimated value of $\rho$ implies a correlation between innovations in the forward rate and its stochastic volatility of $\rho\sigma_W\overline{V_t} = -0.41$.

**Table III.** Parameter Estimates.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\kappa_W$ | 0.021 | (0.000) | $\kappa_J$ | 0.773 | (0.594) | $\theta_W$ | 0.065 | (0.005) | $\theta_J$ | 0.083 | (0.058) |
| $\sigma_W$ | 0.773 | (0.000) | $\sigma_J$ | 2.263 | (0.066) | $\nu_+$ | 0.016 | (0.000) | $\nu_-$ | 0.018 | (0.000) |
| $\gamma^V$ | -0.064 | (0.033) | $\gamma^\nu$ | -0.531 | (0.000) | $\beta_1$ | 0.278 | (0.004) | $\beta_2$ | 0.023 | (0.019) |
| $\beta_3$ | 0.666 | (0.000) | $\beta_4$ | 0.085 | (0.006) | $\rho$ | -0.844 | (0.000) | $\sigma_e^2$ | 0.001 | (0.000) |
| $\mathcal{L}$ | -49,598 | | | | | | | | | | |

Reported are maximum likelihood estimates of the model parameters, their standard errors (in parentheses), and the log-likelihood value denoted by $\mathcal{L}$. We estimate our model on weekly data of implied volatilities for caps and swaptions sampled on Wednesdays, spanning the period August 8, 2007 to August 11, 2010.

As for the jump parameters $\nu^+$ and $\nu^-$, we find that they are highly significant. Hence, during the financial crisis, jumps were an integral part of the forward rate dynamics, both in the forward LIBORs and in the forward swap rates. As the asymmetry between negative and positive jump sizes is small, we also estimated a restricted model with equal jump sizes. We find that, based on a likelihood ratio test, we can reject it in favor of the unrestricted model.

Turning to the risk premium parameter $\gamma^V$ on the stochastic activity rate $V_t$, we get a moderate value of $-0.064$, which, however, is statistically significant. In unreported results using caps data only for the estimation, we were not able to generate an estimate for $\gamma^V$ different from zero. Hence, the inclusion of swaptions data clearly helps to identify the risk premium on stochastic volatility. Compared with the risk premium parameter $\gamma^V$, the parameter for the jump risk premium $\gamma^\nu$ is much larger. Combining the market price of risk coefficient estimates with the extracted risk factors, we can compute the instantaneous risk premium on the two risk factors, i.e., $\sigma_W \gamma^V V_t$ for the instantaneous variance rate and $\sigma_J \gamma^\nu \nu_t$ for the jump activity rate. When we take sample averages for the activity rates, we get a mean risk premium value of -1% and -29%, respectively. Hence, both the magnitude and the time variation of the jump risk premium clearly dominate that of the variance risk premium.

Given the negative values for $\gamma^V$ and $\gamma^\nu$, the mean reversion speed parameters $\kappa_W$ and $\kappa_J$ are both smaller under the terminal forward measure $\mathbb{Q}_n$ than under the historical measure $\mathbb{P}$. Furthermore, the low $\mathbb{Q}_n$-values indicate a high persistence under the pricing measure. Under $\mathbb{P}$, the mean reversion speed of $V_t$ still remains low ($\kappa_W^{\mathbb{P}} = 0.07$), given the low volatility risk premium. In contrast, for $\nu_t$ the mean reversion speed ($\kappa_J^{\mathbb{P}} = 1.97$) suggests a more transient jump activity rate under the historical measure. Finally, the estimated values $\beta_i$ that specify the deterministic volatility function $\lambda(t,T)$ produce a downward sloping volatility curve with $\lim_{t\to T} \lambda(t,T) = 0.34$ and $\lim_{T\to\infty} \lambda(t,T) = 0.08$.

To gain intuition on the role of different parameters on the volatility surface, we shock some of the estimated parameters that are of particular interest, namely, the correlation parameter $\rho$ and the jump sizes $\nu_+$ and $\nu_-$, respectively. As we want to see the parameters' impact across the moneyness, option maturity, and swap tenor dimensions, we focus only on the swaption cube. In Fig. 5, the solid lines in each panel represent the implied volatilities of the swaption cube generated by our model with all parameters set equal to their estimated values and the activity rates set equal to their sample averages. For our analysis, we use swaption implied volatilities with an option maturity of one year and a swap tenor of five years.

We first analyze the impact of parameter shocks along the option maturity dimension. In Fig. 5, Panels A and B display the responses of the swaption implied volatility to changes in $\rho$ and $\nu_\pm$. In Panel A, the implied volatility decreases sharply with maturity when we set $\rho = 0$ (dashed line). For a larger (negative) correlation $\rho = -0.5$ (dash-dotted line), this decline is less dramatic. Hence, the correlation parameter might serve as a mechanism not only to generate negative skew, but also to generate persistency in the implied volatility along the option maturity dimension. Changes in the jump sizes $\nu_\pm$ have only a moderate effect on the implied volatilities as a function of option maturity. Increasing $\nu_+$ to 0.05 and freezing $\nu_-$ at zero (dashed line), the term structure is slightly shifted downward. When we freeze $\nu_+$ at zero and let $\nu_-$ increase to 0.05 (dashed-dotted line), the effect is even smaller. When we symmetrically increase the jump sizes to $\nu_\pm = 0.05$ (dashed line with dots as markers), the term structure across the option maturity dimension remains practically unchanged compared with our current estimates, which are also almost symmetrical, but much smaller (0.016 and 0.018, respectively). Hence, along the option maturity dimension, the asymmetry between positive and negative jumps has a larger effect than the absolute values of the jumps.

Next, we plot the same responses as above, but along the swap tenor dimension (Panels C and D, Fig. 5). We see that a change in the correlation parameter $\rho$ influences equally the implied volatilities at different swap tenors. Hence, changes in $\rho$ lead to a parallel shift of the term structure and not to a shift in the steepness as in Panel A. Changes in the jump sizes $\nu_\pm$ have, to a lesser extent, the same effects. The impact is not as significant as in the case of $\rho$, but it is more pronounced than the effect along the option maturity dimension in Panel B. In Panel D, an asymmetrical shock to the jump sizes leads to a parallel downward shift, but when we symmetrically increase the jump sizes, the term structure exhibits a parallel upward shift. This behavior corresponds to our intuition. When we increase the negative jump parameter, future LIBORs tend to be lower. Through the channel of the leverage effect, lower rates lead to higher volatilities. Similarly, when we increase $\nu_+$ the implicit volatility is lower.

Finally, we plot the responses along the moneyness dimension (Panels E and F, Fig. 5). For the correlation parameter, as expected, we cannot generate a negative skew with $\rho = 0$ (Panel E, dashed line). The implied volatility curve becomes flat. When correlation turns negative, we obtain a negative skew. While shocks in $\nu_{\pm}$ have only moderate impacts along the term structure dimensions, they do have a strong impact along the moneyness dimension (Panel F). When we increase $\nu_{-}$, the implied volatility increases due to the link with the leverage effect and the implied volatility skew becomes more negative (dash-dotted line). When we increase only $\nu_{+}$, volatility decreases and the skew becomes flatter (dashed line). When we symmetrically increase the jumps sizes, the skew moves in an almost parallel fashion. Hence, by splitting up the jumps sizes into negative and positive parts, we can fine-tune the level and steepness of the skew along the moneyness dimension.[19]

To summarize the above analysis, the parameter $\rho$ is essential for all three dimensions of the swaption cube. Along the option maturity dimension, it governs the steepness of the term structure. Along the swap tenor dimension, it influences the level of the term structure and, finally, along the moneyness dimension it is essential to generate a negative skew. In contrast, the jump parameters $\nu_{\pm}$ do not play a significant role along the option maturity dimension. They play a moderate role in the swap tenor dimension. Furthermore, the steepness of the negative skew is driven by the magnitude of the negative shock $\nu_{-}$ relative to $\nu_{+}$.

### VI.2. Pricing errors

In Tables IV to VI we present the root mean squared pricing errors (RMSEs) and the mean pricing errors (MPEs) on caps and swaptions implied volatilities, defined as the difference in percentage points between the model-implied values and the market-implied volatility quotes. Overall, we find that, for intermediate and long maturities, our model performs remarkably well. The cap pricing errors in Table IV indicate that the model's performance suffers mostly at the short end of option

[19] We also estimated the model on a restricted data set, neglecting the information from non-ATM swaptions. We were not able to estimate the jump parameters with statistical significance. Hence, the crucial role of these parameters as an important model device further supports the inclusion of non-ATM swaption quotes into the estimation.

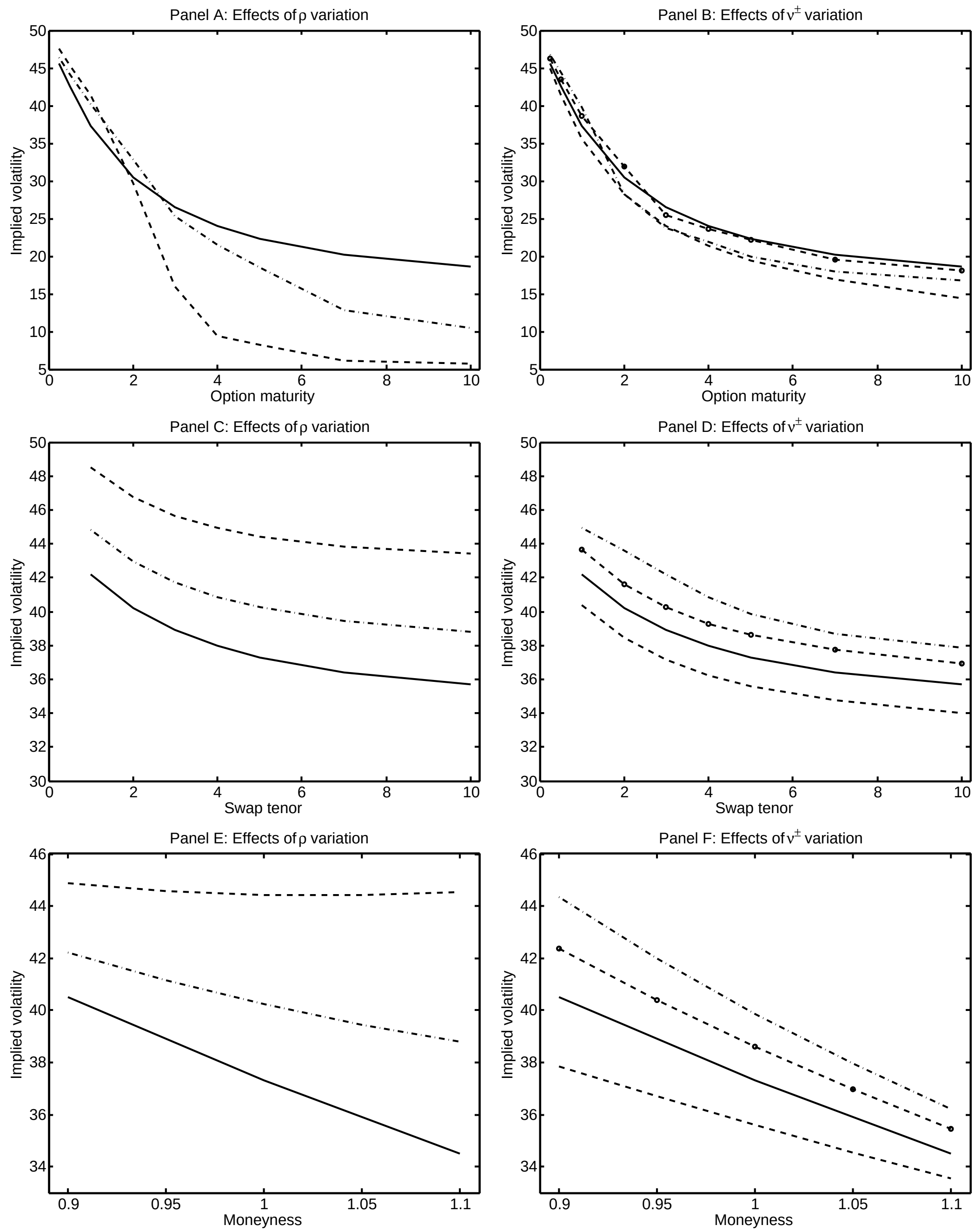


**Figure 5.** Response of swaption implied volatilities. Solid lines in each panel plot implied volatilities in percentage points with all parameters set equal to their estimated values and the activity rates set equal to their sample averages. In Panels A and B we fix the swap tenor to five years and show the responses of the swaption implied volatility to changes in $\rho$ and $\nu^{\pm}$ as a function of the option maturity. In Panels C and D, we do the same plot but as a function of the swap tenor and with an option maturity fixed at one year. In Panels D and E, we fix the swap tenor to five years and the option maturity to one year, and we plot the responses as a function of moneyness. In Panels A, C, and E the dashed lines represent the case $\rho = 0$ and the dash-dotted lines, the case $\rho = -0.5$. In Panels B, D, and F, the dashed lines correspond to $(\nu^+ = 0.05, \nu^- = 0)$, the dash-dotted lines to $(\nu^+ = 0, \nu^- = 0.05)$, and the dashed lines with dots as markers to $(\nu^+ = 0.05, \nu^- = 0.05)$. For Panels A to D we use at-the-money swaptions.

maturities, especially for the one-year maturity. Short maturity contracts are underpriced by the model. However, the pricing performance considerably improves with increasing maturity. For longer maturities, a tendency exists to underprice out-of-the money and overprice in-the-money contracts.

**Table IV.** Pricing errors for the caps market

| | RMSE | | | | | MPE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Maturity* | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 | 0.80 | 0.90 | 1.00 | 1.10 | 1.20 |
| 1 y | 16.86 | 18.62 | 20.15 | 20.30 | 22.03 | -10.73 | -11.08 | -11.95 | -12.32 | -15.96 |
| 2 y | 12.26 | 11.71 | 9.79 | 9.57 | 9.85 | -8.64 | -8.77 | -6.99 | -6.17 | -7.01 |
| 3 y | 8.78 | 6.75 | 4.75 | 4.19 | 4.02 | -6.67 | -5.15 | -3.44 | -2.52 | -2.59 |
| 4 y | 6.47 | 4.29 | 2.25 | 1.66 | 2.03 | -4.81 | -2.98 | -1.40 | -0.48 | -0.06 |
| 5 y | 4.98 | 2.94 | 1.35 | 1.26 | 2.18 | -3.41 | -1.67 | -0.30 | 0.56 | 1.29 |
| 6 y | 4.28 | 2.36 | 1.41 | 1.68 | 2.50 | -2.63 | -0.92 | 0.31 | 1.11 | 1.97 |
| 7 y | 3.91 | 2.16 | 1.71 | 2.08 | 2.71 | -2.07 | -0.38 | 0.74 | 1.48 | 2.25 |
| 8 y | 3.63 | 2.07 | 1.89 | 2.27 | 2.80 | -1.71 | -0.07 | 0.95 | 1.63 | 2.33 |
| 9 y | 3.48 | 2.09 | 2.04 | 2.42 | 2.88 | -1.35 | 0.20 | 1.15 | 1.79 | 2.41 |
| 10 y | 3.37 | 2.15 | 2.18 | 2.54 | 2.95 | -1.06 | 0.42 | 1.31 | 1.91 | 2.47 |

Reported are sample averages of the root mean squared errors (RMSEs) and mean pricing errors (MPEs) for caps implied volatilities, defined as the difference in percentage points between the model-implied values and the market-implied volatility quotes. Each row represents one cap maturity, and columns represent the moneyness of the cap.

For the ATM swaptions implied volatilities in Table V, we observe a similar pattern. The model struggles mostly for short option maturities and short swaption tenors, an observation that also holds true for the non-ATM swaptions in Table VI. However, across moneyness no clear pattern emerges in terms of over- and underpricing as is the case for in-the-money and out-of-the-money caps.

**Table V.** Pricing errors for ATM swaptions

| | *Swap tenor:* | RMSE | | | | | | | MPE | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 y | 2 y | 3 y | 4 y | 5 y | 7 y | 10 y | 1 y | 2 y | 3 y | 4 y | 5 y | 7 y | 10 y |
| *Option* | 3 m | 21.16 | 9.66 | 4.46 | 4.82 | 5.94 | 8.30 | 10.32 | -14.13 | -7.24 | -1.97 | 0.72 | 1.77 | 5.09 | 7.38 |
| *maturity:* | 6 m | 19.53 | 7.93 | 2.58 | 2.60 | 3.78 | 6.17 | 7.89 | -13.67 | -6.05 | -1.45 | 0.90 | 1.93 | 4.35 | 6.10 |
| | 1 y | 12.72 | 5.27 | 1.74 | 1.52 | 2.27 | 3.82 | 4.95 | -8.81 | -3.21 | -0.43 | 0.86 | 1.48 | 2.83 | 3.84 |
| | 2 y | 4.20 | 2.20 | 1.49 | 1.31 | 1.36 | 1.61 | 2.04 | -1.03 | 0.15 | 0.61 | 0.71 | 0.65 | 0.75 | 1.13 |
| | 3 y | 2.34 | 1.75 | 1.38 | 1.18 | 1.13 | 1.22 | 1.41 | 1.33 | 1.16 | 0.86 | 0.51 | 0.16 | -0.05 | -0.03 |
| | 4 y | 2.36 | 1.65 | 1.29 | 1.08 | 1.19 | 1.25 | 1.40 | 1.85 | 1.20 | 0.71 | 0.14 | -0.17 | -0.53 | -0.65 |
| | 5 y | 2.36 | 1.65 | 1.35 | 1.25 | 1.32 | 1.51 | 1.67 | 1.89 | 1.13 | 0.49 | -0.07 | -0.47 | -0.73 | -0.88 |
| | 7 y | 2.10 | 1.53 | 1.36 | 1.32 | 1.32 | 1.54 | 1.80 | 1.44 | 0.73 | 0.28 | -0.16 | -0.44 | -0.74 | -1.07 |
| | 10 y | 1.93 | 1.56 | 1.40 | 1.39 | 1.42 | 1.53 | 1.72 | 1.32 | 0.88 | 0.50 | 0.22 | -0.01 | -0.47 | -0.76 |

Reported are sample averages of the root mean squared errors (RMSEs) and mean pricing errors (MPEs) for ATM swaptions implied volatilities, defined as the difference in percentage points between the model-implied values and the market-implied volatility quotes. Each row represents one swaption maturity, and each column represents one swap tenor.

**Table VI.** Pricing errors for non-ATM swaptions

| *Maturities:* | | RMSE | | | | MPE | | | |
|---|---|---|---|---|---|---|---|---|---|
| *Option* | *Swap* | 0.90 | 0.95 | 1.05 | 1.10 | 0.90 | 0.95 | 1.05 | 1.10 |
| 3 m | 2 y | 1.72 | 10.69 | 7.39 | 13.28 | -0.76 | -7.89 | 2.24 | 4.82 |
| 3 m | 5 y | 11.37 | 12.09 | 5.26 | 12.59 | -8.33 | -8.52 | 1.01 | 9.44 |
| 3 m | 10 y | 9.66 | 11.79 | 4.92 | 9.64 | -7.47 | 5.95 | 0.41 | 6.77 |
| 1 y | 2 y | 9.79 | 5.81 | 2.76 | 7.37 | 6.73 | -3.43 | 1.62 | 6.02 |
| 1 y | 5 y | 5.01 | 6.41 | 1.51 | 5.96 | -3.50 | -3.57 | 0.97 | 4.41 |
| 1 y | 10 y | 5.05 | 3.81 | 1.19 | 4.27 | -3.31 | 2.03 | 0.66 | 3.35 |
| 5 y | 2 y | 3.76 | 1.87 | 1.48 | 2.23 | 2.94 | 1.31 | -0.79 | -1.50 |
| 5 y | 5 y | 1.25 | 2.17 | 1.35 | 1.78 | 0.33 | 1.55 | -0.31 | -1.15 |
| 5 y | 10 y | 1.36 | 1.83 | 1.46 | 1.51 | 0.71 | -1.17 | -0.15 | -0.79 |
| 10 y | 2 y | 1.54 | 1.86 | 1.62 | 2.45 | -0.70 | 1.24 | -0.53 | -1.77 |
| 10 y | 5 y | 1.50 | 2.15 | 1.49 | 2.02 | -0.31 | 1.58 | 0.26 | -1.24 |
| 10 y | 10 y | 1.43 | 1.97 | 1.59 | 1.67 | 0.32 | -1.09 | 0.55 | -0.52 |

Reported are sample averages of the root mean squared errors (RMSEs) and mean pricing errors (MPEs) for non-ATM swaptions implied volatilities, defined as the difference in percentage points between the model-implied values and the market-implied volatility quotes. Each row represents one swaption maturity and one swap tenor. Each column represents one level of moneyness.

The substantially higher pricing errors for the caps and swaptions market at shorter maturities call for further investigation. Ultimately, the caps and swaptions markets must be closely connected, as they both originate from derivatives written on the forward LIBOR. However, during periods of extreme market turmoil, the two markets might exhibit different behaviors due to differences in how the uncertainty regarding the intensified liquidity situation in the interbank market propagates through the caps and swaptions markets. Therefore, we next analyze the behavior of the pricing errors across time to see whether the caps and swaptions market become disintegrated or whether they suffer from the same deficiencies.

In Fig. 6, we plot the time series of RMSE (Panel A) and the MPE (Panel B) for caps implied volatilities. We split the time series into long maturities and short maturities. For the first period of our data sample with the financial crisis already in full swing, the pricing errors in terms of RMSE remain remarkably low. In addition, until October 2008, we do not observe a bias in the model's

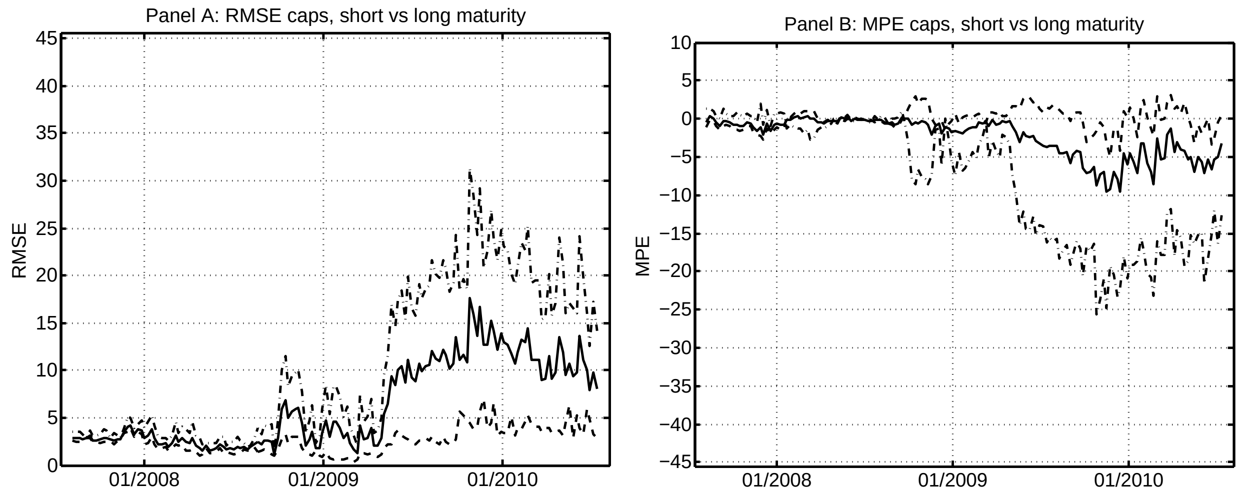


**Figure 6.** Root mean squared error (RMSE) and mean pricing error MPE for caps with different option maturities. Panels A and B show the RMSE and the MPE in percentage points across time for caps implied volatilities of all maturities (solid line), for maturities up to three years (dash-dotted line), and for maturities of four to ten years (dashed line). Data are weekly (Wednesday) spanning our entire data sample August 8, 2007 to August 11, 2010; in total, 158 weeks.

pricing performance with the MPE close to zero. However, the pricing performance deteriorates considerably around April 2009 with substantial underpricing of short maturity contracts. This mispricing remains high until the end of our sample. Interestingly, this period of persistent mispricing of short maturity contracts coincides with the period of high implied volatilities at these maturities (see Fig. 1). Hence, our model suffers when the volatility term structure is unusually steep.

For the swaptions implied volatilities, we observe a similar pattern. Swaptions have two maturity dimensions, the maturity of the swaption and the tenor of the swap. In Fig. 7, Panels A and B, we analyze the pricing errors along the swap tenor dimension. The errors start to increase at the same time as they do for the caps market. The underpricing of the swaptions on short tenor swaps exhibit substantial and systematic underpricing after April 2009, although to a lesser extent than in the caps market. In Fig. 7, Panels D and C, we analyze the pricing errors across option maturities. There seems to be no systematic over- or underpricing. However, if we further split the data and plot the time series of short maturity swaptions with short swap tenors, the underpricing becomes again large and systematic (Panel D, dotted line). As we show in our theoretical derivations, the LIBOR is a special case of a swap rate with $N = 1$. Hence, it is not surprising that the pricing performance of the model for swaptions with short swap tenors resembles the pricing performance of the caps implied volatilities. When we calculate the correlation between the pricing errors of these two time series, we find a correlation of 84% for the RMSE and of 78% for the MPE. Hence, we could argue that both the pricing error and the pricing bias for caps and swaptions with short swap tenors could have a common cause. However, for mid- and long-term contracts, our model performs remarkably well over the whole sample period for both caps and swaptions.

A potential explanation for the model's deteriorating performance toward the end of the sample period could be that increased uncertainty materializes especially for short-term contracts and dries out their liquidity. When we look at the figures, we could argue that the pricing performance of our model deteriorates first around fall 2008 and again in spring 2009. We could link the first deterioration with the spread between LIBOR and the Overnight indexed swap (OIS) rate. Although there was already a sharp rise in the term spreads on August 9, 2007, associated with market concerns related to subprime mortgage market, the spreads further skyrocketed from around 100 basis points to 350 basis points, following the announcement that Lehman Brothers had filed for Chapter 11 bankruptcy in September 2008.

The Lehman bankruptcy certainly had an adverse effect on the pricing performance of our (or any) model. Interestingly, however, the pricing errors increase again in spring 2009, showing a strong persistency. They remain high until the end of our sample period. We conjecture that this systematic bias might be caused by a change in market practice. As of mid–2013, financial market professionals are still coming to grips with the many changes that have occurred in pricing practices since the financial crisis. The basis between LIBOR and OIS, as well as between different parts of the LIBOR curve, blew out dramatically. This prompted some major changes in valuation methodologies. Far from using a single LIBOR curve to discount everything, many dealers started to develop multi-curve

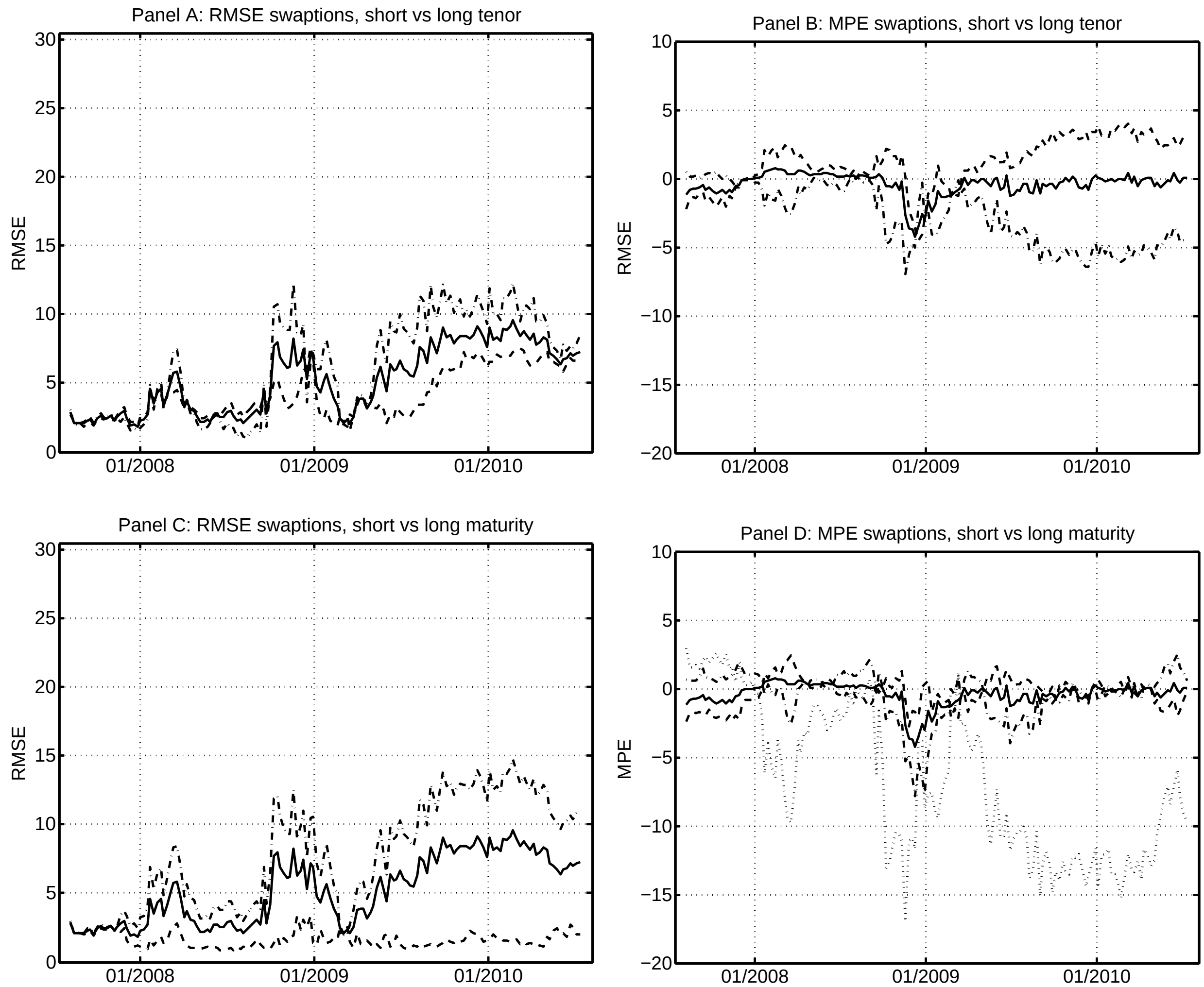


**Figure 7.** Root mean squared error (RMSE) and mean pricing error (MPE) for swaptions with different option maturities and swap tenors. Panels A and B show the RMSE and the MPE in percentage points for swaptions implied volatilities of all tenors (solid line), for tenors up to three years (dash-dotted line) and for tenors of four to ten years (dashed line). In Panels C and D, we plot the RMSE and MPE for swaptions implied volatilities with short option maturities up to two years (dash-dotted line), and for option maturities of three to ten years (dashed line). In Panel D we also plot the MPE for implied volatilities of swaptions with short option maturity and short swap tenor (dotted line). Data are weekly (Wednesday) spanning our entire data sample August 8, 2007 to August 11, 2010; in total, 158 weeks.

valuation models. Single curve and multi-curve approaches can diverge substantially in pricing and risk calculations. If the market adopted a multi-curve approach on a large scale, then our single-curve model might generate a systematic pricing error. As research on multi-curve modeling is still evolving and as of mid-2013, there is no common market practice,[20] a further substantiation of our conjecture is beyond the scope of this paper. Yet, it is an interesting research topic in its own right.

### VI.3. Dynamics of activity rates

In Fig. 8, we plot the extracted state variables $V_t$ and $\nu_t$ from the our estimation over the entire period, August 8, 2007 to August 11, 2010. In Panel A, we plot the activity rate $V_t$. We observe that in 2008 there was a constant increase in this rate with a consolidation at a high level after early 2009. When we look at the jump activity rate $\nu_t$ in Panel B, we see a dramatic increase during the second half of 2009, in its level and its variation. To provide some intuition about the dynamic behavior of the two state variables $V_t$ and $\nu_t$ and their different roles during the financial market crisis, we split our sample into three episodes linked to specific market events.

The first episode is characterized by an increasing volatility $V_t$ and started in the fall of 2007, when the interbank funding market experienced liquidity problems, as indicated by the increasing LIBOR-OIS spread, which reached 108 basis points on December 6, 2007.[21] At that time, the investment bank Lehman Brothers reported large write-downs and subsequently, on March 17, 2008, Bear Stearns collapsed. These events are clearly captured by the first spikes in the activity rates, particularly by the activity rate $V_t$ driving the stochastic volatility of LIBORs. This period was followed by another wave of events in the fall of 2008 that triggered a further increase in the activity rate $V_t$. On September 7, 2008, the government-sponsored entities Fannie Mae and Freddie Mac were placed into conservatorship by the US government. Fannie Mae and Freddie Mac, two key players in the caps and swaptions markets, had large hedges related to their engagement in the

[20] See, e.g., Bianchetti and Carlicchi (2011), among others.

[21] During the financial crisis, the LIBOR-OIS spread was considered a key indicator for the degree of liquidity in the interbank market.

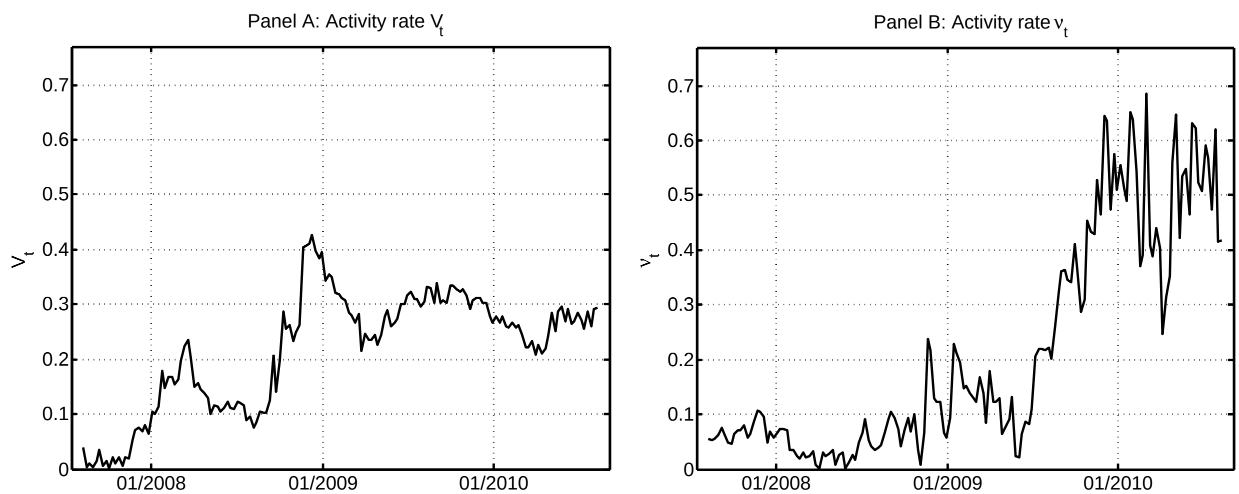


**Figure 8.** Time variation in the state variables. Panel A shows the extracted time variation in the activity rate $V_t$ from the model estimated jointly to the caps and swaptions markets. Panel B shows the corresponding activity rate to the jump component $\nu_t$ in the LIBOR forward rate. Data are weekly (Wednesday) spanning our entire data sample August 8, 2007 to August 11, 2010.

mortgage-backed securities market. On September 15, 2008, Lehman Brothers filed for bankruptcy, which undoubtedly spurred increasing uncertainty in the interbank market. During this period, the activity rates $V_t$ and $\nu_t$ increased substantially.

The second episode starts at the end of 2008. On November 25, 2008, the Fed announced its first quantitative easing program (QE1) to buy \$500 billion in mortgage bonds, effective beginning in January 2009.[22] Even though no MBS had yet been purchased by the Fed, the mere announcement of the QE1 program led to an immediate reduction in the level of the stochastic volatility $V_t$. Hence, the Fed signaled strong and credible backing for mortgage markets in particular and for interest rate markets in general.[23] Subsequently, on January 5, 2009, the Federal Reserve began purchasing

[22] The goal of the QE1 program, as stated by the Federal Reserve in its press release on November 25, 2008, was to "reduce the cost and increase the availability of credit for the purchase of houses, which in turn should support housing markets and foster improved conditions in financial markets more generally." See http://www.federalreserve.gov/newsevents/press/monetary/20081125b.htm.

[23] The study by Hancock and Passmore (2011) is in line with our argumentation. They show that the Fed's announcement reduced mortgage rates by about 85 basis points between November 25 and December 31, 2008.

fixed rate mortgage-backed securities guaranteed by Fannie Mae and Freddie Mac, which further stabilized the volatility in the caps and swaptions markets. However, the average level of $\nu_t$ remained high, which might reflect the market's general unease with the uncertainty surrounding the impact of the Fed's intervention. Then, on May 7, 2009, the results from the US federal stress test of the largest 19 US bank holding companies were announced. The stress test revealed that the 19 companies could potentially lose \$600 billion during 2009 and 2010 if the economy were to follow the adverse scenario considered in the stress test. Subsequently, on May 8 and May 12, respectively, Fannie Mae and Freddie Mac announced large write downs. These events seem to have triggered another period of increasing uncertainty, which hindered $V_t$ from decline further to mid-2008 levels and increased the level of $\nu_t$ to new heights. Hence, market participants could have expected some sudden dramatic changes in future interest rates.

The third and final episode starts toward the end of 2009 and beginning of 2010. On March 31, 2010, the Federal Reserve's mortgage bond buying program ended. The 30-year fixed rate rose to 5.125%. This increase was caused by investors exiting mortgage bond trades before the Fed money was gone. The uncertainty surrounding the end of QE1 caused a large variation in $\nu_t$. This variation was further nourished by the debt problems in Greece. On May 9, 2010, the International Monetary Fund decided to provide financial support to Greece. The increased awareness of a potentially contagious sovereign debt crisis in Europe further increased uncertainty in the interest rate markets and drove global investors into US mortgage and Treasury bonds. Subsequently, the 30-year fixed rate dropped to 4.78%. However, the uncertainty about the future course of interest rates is reflected by the substantial variation in the stochastic arrival rate for jumps $\nu_t$ in the LIBORs during the end of our sample period.

## VII. Conclusion

We introduce a novel time-changed Lévy LIBOR market model. Its design was motivated by a preliminary analysis of the stylized facts about the implied volatilities of the cap surface and the swaption cube. Our model is analytically tractable and yet flexible enough to price caps and swaptions simultaneously. The parsimonious model structure facilitates the identification and stability of the parameter estimates, which is crucial for risk management and hedging.

We find that the incorporation of a jump component and a stochastic volatility factor that is highly correlated with changes in interest rates is crucial for the simultaneous pricing of caps and swaptions. Especially for intermediate and long maturities, we find evidence that the markets for caps and swaptions have been well integrated even during the financial crisis. To explain the volatility skew, we could also have extended the model using a constant elasticity of variance (CEV). However, such an extension is beyond the scope of our paper.

The extension with a CEV-type structure as well as the analysis of the pricing errors during the recent financial crisis could be an interesting avenue for future research. In particular, it would be worthwhile to see what drives the pricing errors of short maturity contracts especially since early 2009. Hence, depending on the availability of data, an important direction for future research based on our results would be the development of models that include additional drivers, such as liquidity or sovereign credit risk, and the potential impact of changing market practices, such as the introduction of multiple discounting curves.

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# Appendix A

### Proof of Proposition 1

From the specification given in Eqs. (4) and (5), we can apply Ito's formula for Lévy processes (e.g., Cont and Tankov, 2004) to obtain the dynamics of the forward LIBOR $L(t, T_j)$ under the $T_{j+1}$-forward measure as follows:

$$\begin{aligned}
\frac{dL(t,T_j)}{L(t,T_j)} &= b(t,T_j,T_{j+1})dt + \frac{1}{2}\lambda^2(t,T_j)dt + \frac{1}{2}V_t^W dt \\
&+ \int_{-\infty}^{0} \left[e^x - 1 - x\right] \pi_{J^-}^{\mathbb{Q}_{j+1}}(dx)\nu_t^J dt + \int_0^{\infty} \left[e^x - 1 - x\right] \pi_{J^+}^{\mathbb{Q}_{j+1}}(dx)\nu_t^J dt \\
&+ \lambda(t,T_j)dB_t^{Q_{j+1}} + \sqrt{V_t^W} dW_t^{Q_{j+1}} + \int_{-\infty}^{0} \left[e^x - 1\right] \left[\mu^-(dt,dx) - \pi_{J^-}^{Q_{j+1}}(x)dx\nu_t^J dt\right] \\
&+ \int_0^{\infty} \left[e^x - 1\right] \left[\mu^+(dt,dx) - \pi_{J^+}^{Q_{j+1}}(x)dx\nu_t^J dt\right]. \qquad \text{(A.1)}
\end{aligned}$$

To ensure that $L(t, T_j)$ is a martingale under the $T_{j+1}$-forward measure, the drift must equal zero, which gives the drift condition in the proposition. ■

### Proof of Proposition 2

The proposition follows by (backward) induction starting from the dynamics of the forward LIBOR under the terminal forward measure $\mathbb{Q}_n$. Consider first the change of measure from $\mathbb{Q}_n$ to $\mathbb{Q}_{n-1}$. The Radon-Nikodym derivative for changing the measure from the $T_n$-forward measure, $\mathbb{Q}_n$, to the

$T_{n-1}$-forward measure, $\mathbb{Q}_{n-1}$, is

$$\frac{d\mathbb{Q}_{n-1}}{d\mathbb{Q}_n}_{|\mathcal{F}_t} = \frac{P(t,T_{n-1})/P(0,T_{n-1})}{P(t,T_n)/P(0,T_n)}. \tag{A.2}$$

The forward LIBORs $L(t,T_{n-1})$ are defined at time $t$ by

$$L(t,T_{n-1}) = \frac{1}{T_n - T_{n-1}}\left[\frac{P(t,T_{n-1})}{P(t,T_n)} - 1\right], \tag{A.3}$$

where $T_n$ denotes the terminal maturity in the LIBOR tenor structure. This can be rewritten for $T_n - T_{n-1} = \delta$ as

$$\frac{P(t,T_{n-1})}{P(t,T_n)} = \delta L(t,T_{n-1}) + 1, \tag{A.4}$$

where $\frac{P(t,T_{n-1})}{P(t,T_n)}$ can be viewed as a forward price process. Consider now the dynamics of this forward price process:

$$\begin{aligned} d\left[\frac{P(t,T_{n-1})}{P(t,T_n)}\right] &= \delta L(t,T_{n-1})\Bigg[\lambda(t,T_{n-1})dB_t^{\mathbb{Q}_n} + \sqrt{V_t^W}dW_t^{\mathbb{Q}_n} \\ &+ \int_{-\infty}^0 [e^x - 1]\left[\mu^-(dt,dx) - \pi_{J^-}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right] \\ &+ \int_0^\infty [e^x - 1]\left[\mu^+(dt,dx) - \pi_{J^+}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right]\Bigg] \qquad \text{(A.5)} \\ &= \left[\frac{P(t,T_{n-1})}{P(t,T_n)}\right]\Bigg[\gamma_\lambda(t,T_{n-1},T_n)dB_t^{\mathbb{Q}_n} + \gamma_V(t,T_{n-1},T_n)dW_t^{\mathbb{Q}_n} \\ &+ \int_{-\infty}^0 \gamma_J(t,T_{n-1},T_n)\left[\mu^-(dt,dx) - \pi_{J^-}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right] \\ &+ \int_0^\infty \gamma_J(t,T_{n-1},T_n)\left[\mu^+(dt,dx) - \pi_{J^+}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right]\Bigg] \qquad \text{(A.6)} \end{aligned}$$

using the fact that $L(t, T_{n-1})$ is a martingale under the measure $\mathbb{Q}_n$ and that

$$\gamma_\lambda(t, T_{n-1}, T_n) = \frac{\delta L(t, T_{n-1})}{\delta L(t, T_{n-1}) + 1} \lambda(t, T_{n-1}), \tag{A.7}$$

$$\gamma_V(t, T_{n-1}, T_n) = \frac{\delta L(t, T_{n-1})}{\delta L(t, T_{n-1}) + 1} \sqrt{V_t^W}, \tag{A.8}$$

and

$$\gamma_J(t, T_{n-1}, T_n) = \frac{\delta L(t, T_{n-1})}{\delta L(t, T_{n-1}) + 1} \left[e^x - 1\right]. \tag{A.9}$$

Simplifying the above by putting $F_B(t, T_{n-1}, T_n) = \frac{P(t,T_{n-1})}{P(t,T_n)}$, and

$$\begin{aligned} H(t, T_n) &= \int_0^t \gamma_\lambda(s, T_{n-1}, T_n) dB_s^{\mathbb{Q}_n} + \gamma_V(s, T_{n-1}, T_n) dW_s^{\mathbb{Q}_n} \\ &+ \int_0^t \int_{-\infty}^0 \gamma_J(t, T_{n-1}, T_n) \left[\mu^-(dt, dx) - \pi_{J^-}^{\mathbb{Q}_n}(x) dx \nu_t^J dt\right] \\ &+ \int_0^t \int_0^\infty \gamma_J(t, T_{n-1}, T_n) \left[\mu^+(dt, dx) - \pi_{J^+}^{\mathbb{Q}_n}(x) dx \nu_t^J dt\right], \end{aligned} \tag{A.10}$$

the stochastic differential equation can be reformulated as

$$dF_B(t, T_{n-1}, T_n) = F_B(t, T_{n-1}, T_n) dH(t, T_n) \tag{A.11}$$

from which we know the solution is the Doléans–Dade stochastic exponential given by

$$F_B(t, T_{n-1}, T_n) = F_B(0, T_{n-1}, T_n) \mathcal{E}\left[H(t, T_n)\right]. \tag{A.12}$$

The term $T_n$ in $H(t, T_n)$ indicates that the stochastic exponential is taken under the $T_{n-1}$-forward measure. Therefore,

$$\frac{P(t, T_{n-1})/P(0, T_{n-1})}{P(t, T_n)/P(0, T_n)} = \mathcal{E}\left[H(t, T_n)\right]. \tag{A.13}$$

We observe that the likelihood process we are looking for to change measure from the $T_n$-forward measure, $\mathbb{Q}_n$, to the $T_{n-1}$-forward measure, $\mathbb{Q}_{n-1}$, is defined by the stochastic exponential of the process $H(t,T_n)$ (which by definition is a $\mathbb{Q}_n$-martingale due to the martingale preserving property; see Proposition 8.23 in Cont and Tankov, 2004). Writing the measure transformation in more familiar terms,

$$\begin{aligned}
\frac{d\mathbb{Q}_{n-1}}{d\mathbb{Q}_n}_{|\mathcal{F}_t} &= \mathcal{E}\Bigg[\int_0^t \gamma_\lambda(s,T_{n-1},T_n)dB_s^{\mathbb{Q}_n} + \gamma_V(s,T_{n-1},T_n)dW_s^{\mathbb{Q}_n} \\
&+ \int_0^t\int_{-\infty}^0 \gamma_J(t,T_{n-1},T_n)\left[\mu^-(dt,dx) - \pi_{J^-}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right] \\
&+ \int_0^t\int_0^\infty \gamma_J(t,T_{n-1},T_n)\left[\mu^+(dt,dx) - \pi_{J^+}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right]\Bigg], \qquad \text{(A.14)}
\end{aligned}$$

for all $t \in [0,T_{n-1}]$. We can now identify the Girsanov kernel in the measure transformation related to the Brownian motion part, which allows us to change from one forward measure to another:

$$dB_t^{\mathbb{Q}_{n-1}} = dB_t^{\mathbb{Q}_n} - \langle dB_t^{\mathbb{Q}_n}, \gamma_\lambda(t,T_{n-1},T_n)dB_t^{\mathbb{Q}_n}\rangle = dB_t^{\mathbb{Q}_n} - \gamma_\lambda(t,T_{n-1},T_n)dt \qquad \text{(A.15)}$$

and

$$dW_t^{\mathbb{Q}_{n-1}} = dW_t^{\mathbb{Q}_n} - \langle dW_t^{\mathbb{Q}_n}, \gamma_V(t,T_{n-1},T_n)dW_t^{\mathbb{Q}_n}\rangle = dW_t^{\mathbb{Q}_n} - \gamma_V(t,T_{n-1},T_n)dt, \quad \text{(A.16)}$$

where $\langle\cdot,\cdot\rangle$ denotes the covariance. Moreover, we observe that the change of intensity of the jump component (e.g., Bjork, Kabanov, and Runggaldier, 1997) takes the form

$$\pi_J^{\mathbb{Q}_{n-1}}\nu^J = \left[1+\gamma_J(t,T_{n-1},T_n)\right]\pi_J^{\mathbb{Q}_n}\nu^J, \qquad \text{(A.17)}$$

for $J = \{J^-, J^+\}$. Consider next the measure change $\mathbb{Q}_{n-1}$ to $\mathbb{Q}_{n-2}$. The Radon–Nikodym derivative for changing the measure from the $T_{n-1}$-forward measure, $\mathbb{Q}_{n-1}$, to the $T_{n-2}$-forward measure,

$\mathbb{Q}_{n-2}$, is

$$\frac{d\mathbb{Q}_{n-2}}{d\mathbb{Q}_{n-1}}_{|\mathcal{F}_t} = \frac{P(t,T_{n-2})/P(0,T_{n-2})}{P(t,T_{n-1})/P(0,T_{n-1})}. \tag{A.18}$$

The dynamics of the forward price process is given by

$$\begin{aligned} d\left[\frac{P(t,T_{n-2})}{P(t,T_{n-1})}\right] &= \left[\frac{P(t,T_{n-2})}{P(t,T_{n-1})}\right]\Big[\gamma_\lambda(t,T_{n-2},T_{n-1})dB_t^{\mathbb{Q}_{n-1}} + \gamma_V(t,T_{n-2},T_{n-1})dW_t^{\mathbb{Q}_{n-1}} \\ &+ \int_{-\infty}^{0}\gamma_J(t,T_{n-2},T_{n-1})\left[\mu^-(dt,dx) - \pi_{J^-}^{\mathbb{Q}_{n-1}}(x)dx\nu_t^J dt\right] \\ &+ \int_{0}^{\infty}\gamma_J(t,T_{n-2},T_{n-1})\left[\mu^+(dt,dx) - \pi_{J^+}^{\mathbb{Q}_{n-1}}(x)dx\nu_t^J dt\right]\Big], \end{aligned} \tag{A.19}$$

where

$$\gamma_\lambda(t,T_{n-2},T_{n-1}) = \frac{\delta L(t,T_{n-2})}{\delta L(t,T_{n-2})+1}\lambda(t,T_{n-2}), \tag{A.20}$$

$$\gamma_V(t,T_{n-2},T_{n-1}) = \frac{\delta L(t,T_{n-2})}{\delta L(t,T_{n-2})+1}\sqrt{V_t^W}, \tag{A.21}$$

and

$$\gamma_J(t,T_{n-2},T_{n-1}) = \frac{\delta L(t,T_{n-2})}{\delta L(t,T_{n-2})+1}\left[e^x - 1\right]. \tag{A.22}$$

Similarly to the above, we end up with

$$\frac{P(t,T_{n-2})/P(0,T_{n-2})}{P(t,T_{n-1})/P(0,T_{n-1})} = \mathcal{E}\left[H(t,T_{n-1})\right], \tag{A.23}$$

where

$$\begin{aligned}
H(t,T_{n-1}) &= \int_0^t \gamma_\lambda(s,T_{n-2},T_{n-1})dB_s^{\mathbb{Q}_{n-1}} + \gamma_V(s,T_{n-2},T_{n-1})dW_s^{\mathbb{Q}_{n-1}} \\
&+ \int_0^t \int_{-\infty}^0 \gamma_J(t,T_{n-2},T_{n-1}) \left[\mu^-(dt,dx) - \pi_{J^-}^{\mathbb{Q}_{n-1}}(x)dx\nu_t^J dt\right] \\
&+ \int_0^t \int_0^\infty \gamma_J(t,T_{n-2},T_{n-1}) \left[\mu^+(dt,dx) - \pi_{J^+}^{\mathbb{Q}_{n-1}}(x)dx\nu_t^J dt\right], \qquad \text{(A.24)}
\end{aligned}$$

and again we observe that the likelihood process we are looking for to change measure from the $T_{n-1}$-forward measure, $\mathbb{Q}_{n-1}$, to the $T_{n-2}$-forward measure, $\mathbb{Q}_{n-2}$, which is defined by the stochastic exponential of the process $H(t,T_{n-1})$, which by definition is a $\mathbb{Q}_{n-1}$-martingale. As above, we can write the measure transformation in more familiar terms as

$$\begin{aligned}
\frac{d\mathbb{Q}_{n-2}}{d\mathbb{Q}_{n-1}}_{|\mathcal{F}_t} &= \mathcal{E}\left[\int_0^t \gamma_\lambda(s,T_{n-2},T_{n-1})dB_s^{\mathbb{Q}_{n-1}} + \gamma_V(s,T_{n-2},T_{n-1})dW_s^{\mathbb{Q}_{n-1}}\right. \\
&+ \int_0^t \int_{-\infty}^0 \gamma_J(t,T_{n-2},T_{n-1}) \left[\mu^-(dt,dx) - \pi_{J^-}^{\mathbb{Q}_{n-1}}(x)dx\nu_t^J dt\right] \\
&+ \left.\int_0^t \int_0^\infty \gamma_J(t,T_{n-2},T_{n-1}) \left[\mu^+(dt,dx) - \pi_{J^+}^{\mathbb{Q}_{n-1}}(x)dx\nu_t^J dt\right]\right] \qquad \text{(A.25)}
\end{aligned}$$

for all $t \in [0, T_{n-2}]$. We can now identify the Girsanov kernel in the measure transformation related to the Brownian motion part, which allows us to change from one forward measure to another:

$$\begin{aligned}
dB_t^{\mathbb{Q}_{n-2}} &= dB_t^{\mathbb{Q}_{n-1}} - \langle dB_t^{\mathbb{Q}_{n-1}}, \gamma_\lambda(t,T_{n-2},T_{n-1})dB_t^{\mathbb{Q}_{n-1}}\rangle \qquad \text{(A.26)} \\
&= dB_t^{\mathbb{Q}_{n-1}} - \gamma_\lambda(t,T_{n-2},T_{n-1})dt \qquad \text{(A.27)} \\
&= dB_t^{\mathbb{Q}_n} - \sum_{k=1}^{2} \gamma_\lambda\left[t,T_{n-k},T_{n+1-k}\right]dt \qquad \text{(A.28)}
\end{aligned}$$

and

$$\begin{aligned}
dW_t^{\mathbb{Q}_{n-2}} &= dW_t^{\mathbb{Q}_{n-1}} - \langle dW_t^{\mathbb{Q}_{n-1}}, \gamma_V(t, T_{n-2}, T_{n-1}) dW_t^{\mathbb{Q}_{n-1}} \rangle && \text{(A.29)} \\
&= dW_t^{\mathbb{Q}_{n-1}} - \gamma_V(t, T_{n-2}, T_{n-1}) dt && \text{(A.30)} \\
&= dW_t^{\mathbb{Q}_n} - \sum_{k=1}^{2} \gamma_V \left[t, T_{n-k}, T_{n+1-k}\right] dt. && \text{(A.31)}
\end{aligned}$$

The change of intensity of the jump component takes the form

$$\begin{aligned}
\pi_J^{\mathbb{Q}_{n-2}} \nu^J &= \left[1 + \gamma_J(t, T_{n-2}, T_{n-1})\right] \pi_J^{\mathbb{Q}_{n-1}} \nu^J && \text{(A.32)} \\
&= \prod_{k=1}^{2} \left[1 + \gamma_J(t, T_{n-k}, T_{n+1-k})\right] \pi_J^{\mathbb{Q}_n} \nu^J && \text{(A.33)}
\end{aligned}$$

for $J = \{J^-, J^+\}$. Continuing the same procedure along the entire tenor structure, we can, for $j = 2, \ldots, n$, summarize the forward measure transformations that define the family of spanning forward LIBORs related to the terminal measure (i.e., the $\mathbb{Q}_n$ measure):

$$\begin{aligned}
dB_t^{\mathbb{Q}_{j-1}} &= dB_t^{\mathbb{Q}_n} - \sum_{k=1}^{n+1-j} \gamma_\lambda \left[t, T_{n-k}, T_{n+1-k}\right] dt, && \text{(A.34)} \\
dW_t^{\mathbb{Q}_{j-1}} &= dW_t^{\mathbb{Q}_n} - \sum_{k=1}^{n+1-j} \gamma_V \left[t, T_{n-k}, T_{n+1-k}\right] dt, && \text{(A.35)}
\end{aligned}$$

and

$$\pi_J^{\mathbb{Q}_{j-1}} \nu_t^J = \prod_{k=1}^{n+1-j} \left[1 + \gamma_J(t, T_{n-k}, T_{n+1-k})\right] \pi_J^{\mathbb{Q}_n} \nu^J, \qquad \text{(A.36)}$$

where

$$\begin{aligned}
\gamma_\lambda(t, T_{n-k}, T_{n+1-k}) &= \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \lambda(t, T_{n-k}), && \text{(A.37)} \\
\gamma_V(t, T_{n-k}, T_{n+1-k}) &= \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \sqrt{V_t^W}, && \text{(A.38)}
\end{aligned}$$

and

$$\gamma_J(t, T_{n-k}, T_{n+1-k}) = \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \left[e^x - 1\right] \tag{A.39}$$

for the jump components, where $J = \{J^-, J^+\}$, respectively, as given in the proposition. ■

### Proof of Proposition 3

We can proceed similarly to the proof of Proposition 2. Consider first $\mathbb{Q}_n$ to $\mathbb{Q}_{n-1}$:

$$\begin{aligned} d\widetilde{W}^{\mathbb{Q}_{n-1}} &= d\widetilde{W}^{\mathbb{Q}_n} - \langle d\widetilde{W}^{\mathbb{Q}_n}, \gamma_V(t, T_{n-1}, T_n)\sqrt{V_t^W} dW^{\mathbb{Q}_n}\rangle && \text{(A.40)} \\ &= d\widetilde{W}^{\mathbb{Q}_n} - \frac{\delta L(t, T_{n-1})}{\delta L(t, T_{n-1}) + 1}\sqrt{V_t^W}\rho dt. && \text{(A.41)} \end{aligned}$$

Now consider $\mathbb{Q}_{n-1}$ to $\mathbb{Q}_{n-2}$:

$$\begin{aligned} d\widetilde{W}^{\mathbb{Q}_{n-2}} &= d\widetilde{W}^{\mathbb{Q}_{n-1}} - \langle d\widetilde{W}^{\mathbb{Q}_{n-1}}, \gamma_V(t, T_{n-2}, T_{n-1})\sqrt{V_t^W} dW^{\mathbb{Q}_{n-1}}\rangle && \text{(A.42)} \\ &= d\widetilde{W}^{\mathbb{Q}_{n-1}} - \frac{\delta L(t, T_{n-2})}{\delta L(t, T_{n-2}) + 1}\sqrt{V_t^W}\rho dt, && \text{(A.43)} \end{aligned}$$

which can be related to the terminal measure, $\mathbb{Q}_n$:

$$d\widetilde{W}^{\mathbb{Q}_{n-2}} = d\widetilde{W}^{\mathbb{Q}_n} - \sum_{k=1}^{2} \frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1}\sqrt{V_t^W}\rho dt. \tag{A.44}$$

Continuing the same procedure along the entire tenor structure, we can, for $j = 2, \ldots, n$, summarize the forward measure transformations that define the family of measure changes for the stochastic volatility process by their relation to the terminal forward measure, $\mathbb{Q}_n$, as stated in the proposition.

■

## Proof of Proposition 4

To prove Proposition 4, we need Lemma A.1.

**Lemma A.1.** *The co-sliding forward swap rate can be approximated by*

$$R_j^N(t) \approx R_j^N(0) \exp\left[\int_0^t \sum_{k=j}^{j+N-1} \tilde{\omega}_k(0) dX_s + \mathit{drift}\right], \tag{A.45}$$

*where $X_s$ is the Lévy process under the terminal forward LIBOR measure in Eq.* (5) *and* $\tilde{\omega}_k(0) = \frac{\omega_k(0)L(0,T_k)}{R_j^N(0)}$ *with* $\omega_k(0) = \frac{P(0,T_{k+1})}{\sum_{k=j+1}^{j+N} P(0,T_k)}$.

**Proof.** We can represent the co-sliding forward swap rate as a weighted average of spanning forward LIBORs:

$$R_j^N(t) \;=\; \frac{P(t,T_j) - P(t,T_{j+N})}{\delta \sum_{k=j+1}^{j+N} P(t,T_k)} = \sum_{k=j}^{j+N-1} \omega_k(t) L(t,T_k) \tag{A.46}$$

where the weights are given by $\omega_k(t) = \frac{P(t,T_{k+1})}{\sum_{k=j+1}^{j+N} P(t,T_k)}$. To obtain analytical tractability, we freeze the weights $\omega_k(t)$ at time $t = 0$ and we use the approximation $ye^x \approx y + yx$ for $x$ small.[24] Then,

$$\begin{aligned}
R_j^N(t) &\approx \sum_{k=j}^{j+N-1} \omega_k(0) L(t,T_k) = \sum_{k=j}^{j+N-1} \omega_k(0) L(0,T_k) \exp\left[\int_0^t dX_s + \mathit{drift}\right] && \text{(A.47)}\\
&\approx \sum_{k=j}^{j+N-1} \omega_k(0) L(0,T_k) + \sum_{k=j}^{j+N-1} \omega_k(0) L(0,T_k) \left[\int_0^t dX_s + \mathit{drift}\right] && \text{(A.48)}\\
&= R_j^N(0) + R_j^N(0) \left[\int_0^t \sum_{k=j}^{j+N-1} \frac{\omega_k(0) L(0,T_k)}{R_j^N(0)} dX_s + \mathit{drift}\right] && \text{(A.49)}\\
&\approx R_j^N(0) \exp\left[\int_0^t \sum_{k=j}^{j+N-1} \tilde{\omega}_k(0) dX_s + \mathit{drift}\right], && \text{(A.50)}
\end{aligned}$$

[24] Because the precise drift specification is not relevant for pricing, we do not write it out explicitly.

where $X_t^{\mathbb{Q}_n}$ is the Lévy process under the terminal forward measure given in Eq. (5) and $\tilde{\omega}_k(0) = \frac{\omega_k(0)L(0,T_k)}{R_j^N(0)}$. □

Lemma A.1 allows us to approximately model the co-sliding forward swap rate as an exponential of the Lévy process $X_s$ under the terminal forward LIBOR measure. However, for pricing purposes, it is convenient to formulate the forward swap rate dynamics under the appropriate terminal co-sliding forward swap measures under which the forward swap rate is a martingale. By doing so, we are consistent with the Black (1976) model, which is currently market practice for valuing swaption derivatives.

Similar to the LMM, we can construct an entire family of co-sliding forward swap rates by backward induction starting from the terminal co-sliding forward swap measure for a given $N$. However, before we can perform the backward construction of the family of co-sliding swap rates, we first need to establish the change of measure, which takes us from the terminal forward LIBOR measure $\mathbb{Q}_n$ to the terminal co-sliding forward swap measure $\mathbb{Q}_{n-N+1}^N$ for a given $N$. $\mathbb{Q}_n$ coincides with $\mathbb{Q}_{n-N+1}^N$ for $N = 1$. Hence, we get Lemma A.2.

**Lemma A.2.** *Under the approximation of the forward swap rate in Eq.* (A.45)*, the Radon-Nikodym derivative of* $\mathbb{Q}_{n-N+1}^N$ *with respect to* $\mathbb{Q}_n$ *is*

$$\begin{aligned}
\frac{d\mathbb{Q}_{n-N+1}^N}{d\mathbb{Q}_n}_{|\mathcal{F}_t} &\approx \mathcal{E}\Bigg[\int_0^t \varphi_1 dB_s^{\mathbb{Q}_n} + \int_0^t \varphi_2 \sqrt{V_s^W} dW_s^{\mathbb{Q}_n} \\
&+ \int_0^t \int_{-\infty}^0 (e^{\varphi_2 x} - 1)\left[\mu^-(dt,dx) - \pi_{J^-}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right] \\
&+ \int_0^t \int_0^\infty (e^{\varphi_2 x} - 1)\left[\mu^+(dt,dx) - \pi_{J^+}^{\mathbb{Q}_n}(x)dx\nu_t^J dt\right]\Bigg], \qquad \text{(A.51)}
\end{aligned}$$

*where* $\mathcal{E}(\cdot)$ *is the Doléans-Dade exponential and*

$$\varphi_1 = \sum_{k=n-N}^{n-1}\left[\frac{P(0,T_{n-N})}{P(0,T_{n-N}) - P(0,T_n)}\frac{\delta L(0,T_k)}{1+\delta L(0,T_k)} - \tilde{\omega}_k(0)\right]\lambda(s,T_k) \qquad \text{(A.52)}$$

*and*

$$\varphi_2 \quad = \quad \frac{P(0,T_{n-N})}{P(0,T_{n-N})-P(0,T_n)} \sum_{k=n-N}^{n-1} \frac{\delta L(0,T_k)}{1+\delta L(0,T_k)} - 1. \tag{A.53}$$

**Proof.** Consider the Radon-Nikodym derivative for some $j = 0, \ldots, n-N$ defined by the following quantity,

$$\frac{d\mathbb{Q}_{j+1}^N}{d\mathbb{Q}_n}_{|\mathcal{F}_t} = \frac{S_t(T_j,T_{j+N})}{P(t,T_n)} \frac{P(0,T_n)}{S_0(T_j,T_{j+N})}, \tag{A.54}$$

which by definition is a $\mathbb{Q}_n$-martingale. We first fix $j = n-N$ for a given $N$ and look at the first term in Eq. (A.54):

$$\frac{S_t(T_{n-N},T_n)}{P(t,T_n)} = \frac{1}{R_{n-N}^N(t)} \left[ \frac{P(t,T_{n-N})}{P(t,T_n)} - 1 \right] = \frac{1}{R_{n-N}^N(t)} \left[ \prod_{k=n-N}^{n-1} (1+\delta L(t,T_k)) - 1 \right]. \tag{A.55}$$

By inserting our dynamics of the forward LIBOR and the approximate dynamics of the forward swap rate in Eq. (A.45), we get, under $\mathbb{Q}_n$,

$$\begin{aligned} \frac{S_t(T_{n-N},T_n)}{P(t,T_n)} \quad &\approx \quad \frac{1}{R_{n-N}^N(0)} \exp\left[ -\int_0^t \sum_{k=n-N}^{n-1} \tilde{\omega}_k(0) dX_s + \mathit{drift} \right] \\ &\times \quad \left[ \prod_{k=n-N}^{n-1} \left[ 1 + \delta L(0,T_k) \exp\left[ \int_0^t dX_s + \mathit{drift} \right] \right] - 1 \right] \qquad \text{(A.56)} \\ &\approx \quad \frac{1}{R_{n-N}^N(0)} \frac{P(0,T_{n-N})-P(0,T_n)}{P(0,T_n)} \\ &\times \quad \exp\left[ \sum_{k=n-N}^{n-1} \int_0^t \left[ \frac{P(0,T_{n-N}) \frac{\delta L(0,T_k)}{1+\delta L(0,T_k)}}{P(0,T_{n-N})-P(0,T_n)} - \tilde{\omega}_k(0) \right] dX_s + \mathit{drift} \right], \qquad \text{(A.57)} \end{aligned}$$

by relying (twice) on the approximation $1+\kappa\exp(x) \approx (1+\kappa)\exp(\frac{\kappa}{1+\kappa}x)$ for $x$ small.[25] Inserting

[25] See, e.g., Kluge (2005, p. 73).

the definition of the process $dX_s$ under the measure $\mathbb{Q}_n$ proves the proposition. $\square$ Applying the above change of measure result, we get the dynamics of the Lévy components in Proposition 4. $\blacksquare$

## Proof of Proposition 5

For simplicity, we fix $t = 0$. Under some technical conditions, we obtain the Fourier transform of the cap as

$$\chi(z) = \int_{-\infty}^{\infty} e^{izk} C_0(k, T_j) dk = \delta P(0, T_{j+1}) \mathbb{E}_0^{Q_{j+1}} \left[ \int_{-\infty}^{Y_{T_j}} e^{izk} \left[ e^{Y_{T_j}} - e^k \right] dk \right] \tag{A.58}$$

$$= \delta P(0, T_{j+1}) \mathbb{E}_0^{Q_{j+1}} \left[ \frac{e^{(1+iz)Y_{T_j}}}{iz} - \frac{e^{(1+iz)Y_{T_j}}}{1+iz} \right] = \delta P(0, T_{j+1}) \frac{\phi_{Y_{T_j}}(z-i)}{(iz)(iz+1)}, \tag{A.59}$$

where $\phi_{Y_{T_j}}(\cdot)$ denotes the characteristic function of $Y_{T_j}$.[26] Then the option value can be calculated using the Fourier inversion formula:

$$C_0(k, T_j) = \frac{1}{2} \int_{-iz_i-\infty}^{-iz_i+\infty} e^{-izk} \chi(z) dz = \frac{e^{-z_i k}}{\pi} \int_0^{\infty} e^{-iz_r k} \chi(z_r - iz_i) dz_r \tag{A.60}$$

$$= \frac{e^{-z_i k}}{\pi} \delta P(0, T_{j+1}) \int_0^{\infty} e^{-iz_r k} \frac{\phi_{Y_{T_j}}(z_r - iz_i - i)}{(iz_r + z_i)(iz_r + z_i + 1)} dz_r. \tag{A.61}$$

[26] See, e.g., Wu (2008).

The complex valued measure as developed in Carr and Wu (2004) allows us to write the characteristic function $\phi_{Y_{T_j}}(u) = \mathbb{E}_0^{Q_{j+1}}(\exp(iuY_{T_j}))$ as

$$\begin{aligned}
\phi_{Y_{T_j}}(u) &= \mathbb{E}_0^{Q_{j+1}}\left[\exp\left(iu\left[\ln L(0,T_j) + \int_0^{T_j} b(s,T_j,T_{j+1})ds + \int_0^{T_j} dX_s^{Q_{j+1}}\right]\right)\right] && \text{(A.62)}\\
&= L(0,T_j)^{iu}\mathbb{E}_0^{\mathbb{Q}_{j+1}}\left[\exp\left(iu\left[\int_0^{T_j} b(s,T_j,T_{j+1})ds + \int_0^{T_j} dX_s^{Q_{j+1}}\right]\right)\right] && \text{(A.63)}\\
&= L(0,T_j)^{iu}\mathbb{E}_0^{Q_{j+1}}\left[\exp\left(iu\left[B^{Q_{j+1}}_{\mathcal{T}^B_{T_j}} - \frac{1}{2}\mathcal{T}^B_{T_j}\right] + iu\left[W^{Q_{j+1}}_{\mathcal{T}^W_{T_j}} - \frac{1}{2}\mathcal{T}^W_{T_j}\right]\right.\right.\\
&+ \left.\left. iu\left[J^{Q_{j+1}}_{\mathcal{T}^J_{T_j}} - k_J(1)\mathcal{T}^J_{T_j}\right]\right)\right] && \text{(A.64)}\\
&= L(0,T_j)^{iu}\mathbb{E}_0^{\mathbb{M}}\left[\exp\left(-\psi_B^{\mathbb{Q}_{j+1}}(u)\mathcal{T}^B_{T_j} - \psi_W^{\mathbb{Q}_{j+1}}(u)\mathcal{T}^W_{T_j} - \psi_J^{\mathbb{Q}_{j+1}}(u)\mathcal{T}^J_{T_j}\right)\right], && \text{(A.65)}
\end{aligned}$$

where $\psi_B^{\mathbb{Q}_{j+1}}(u)$, $\psi_W^{\mathbb{Q}_{j+1}}(u)$, and $\psi_J^{\mathbb{Q}_{j+1}}(u)$ denote the characteristic exponents of the convexity-adjusted Lévy components and where the time changes are given by

$$\mathcal{T}^B_{T_j} = \int_0^{T_j} \lambda^2(s,T_j)ds, \qquad \mathcal{T}^W_{T_j} = \int_0^{T_j} V_s^W ds, \qquad \mathcal{T}^J_{T_j} = \int_0^{T_j} \nu_s^J ds. \tag{A.66}$$

The change to the measure $\mathbb{M}$ in Eq. (A.65) is given by the complex valued exponential $\mathbb{Q}_{j+1}$-martingale,

$$\begin{aligned}
\frac{d\mathbb{M}}{d\mathbb{Q}_{j+1}}_{|\mathcal{F}_t} &= \exp\left[iu\left[B^{Q_{j+1}}_{\mathcal{T}^B_{T_j}} - \frac{1}{2}\mathcal{T}^B_{T_j}\right] + \psi_B^{\mathbb{Q}_{j+1}}(u)\mathcal{T}^B_{T_j} + iu\left[W^{Q_{j+1}}_{\mathcal{T}^W_{T_j}} - \frac{1}{2}\mathcal{T}^W_{T_j}\right] + \psi_W^{\mathbb{Q}_{j+1}}(u)\mathcal{T}^W_{T_j}\right.\\
&+ \left. iu\left[J^{Q_{j+1}}_{\mathcal{T}^J_{T_j}} - k_J(1)\mathcal{T}^J_{T_j}\right] + \psi_J^{\mathbb{Q}_{j+1}}(u)\mathcal{T}^J_{T_j}\right] && \text{(A.67)}
\end{aligned}$$

for $j = 0, 1 \ldots, n-1$, where $k_J(1)$ represents the cumulant exponent of the Lévy jump component.[27] The advantage of the representation of the characteristic exponent in Eq. (A.65) is that we can decompose the problem. First, we need to find the characteristic exponents of the Lévy components prior to the time change. Second, we need to find the characteristic exponents of the time changes. If all of these parts are analytically tractable, then the characteristic exponent of the time-changed Lévy process is tractable. We start by looking at the terms $\psi_B^{\mathbb{Q}_{j+1}}(u)$ and $\psi_W^{\mathbb{Q}_{j+1}}(u)$ in Eq. (A.65), which are the characteristic exponents of the two convexity-adjusted continuous Lévy components $B_t - \frac{1}{2}t$ and $W_t - \frac{1}{2}t$, respectively. Because they do not depend on $T_{j+1}$, we simplify the notation and get

$$\psi_B(u) \quad = \quad \psi_W(u) = \frac{1}{2}\left[iu + u^2\right]. \tag{A.68}$$

Next, we determine the characteristic exponent $\psi_J^{\mathbb{Q}_{j+1}}(u)$ of the convexity-adjusted jump component $J_t - k_J(1)t$. For analytical tractability, we adopt the commonly used freezing of coefficients approximation:

$$\frac{\delta L(t, T_{n-k})}{\delta L(t, T_{n-k}) + 1} \approx \frac{\delta L(0, T_{n-k})}{\delta L(0, T_{n-k}) + 1}. \tag{A.69}$$

Then, to determine the approximate characteristic exponent under the appropriate forward measure, $\mathbb{Q}_{j+1}$, we have to compute the integral

$$\psi_J^{\mathbb{Q}_{j+1}}(u) \quad \approx \quad \int_{\mathbb{R}_0} \left[1 - e^{iux}\right] \prod_{k=1}^{n-1-j} \left[1 + \frac{\delta L(0, T_{n-k})}{\delta L(0, T_{n-k}) + 1} \left[e^x - 1\right]\right] \pi_J^{\mathbb{Q}_n} dx \tag{A.70}$$

for $j = 0, 1 \ldots, n-1$. Starting from the terminal measure $\mathbb{Q}_n$ (i.e., for $j = n-1$) and given the density describing the arrival rate of jumps, we can calculate the characteristic exponent of the

[27] $W^{\mathbb{Q}_{j+1}}_{\mathcal{T}^W_{T_j}}$ denotes the standard Brownian motion $W$ under the $T_{j+1}$-forward measure and subordinated to the stochastic time change defined by $\mathcal{T}^W_{T_j}$.

convexity-adjusted jump component:

$$\psi_J^{\mathbb{Q}_n}(u) = \int_{\mathbb{R}_0} \left[1 - e^{iux}\right] \pi_J^{\mathbb{Q}_n} dx - k_J(1) \tag{A.71}$$

$$= \ln\left[(1 - iu\nu_{J+})(1 + iu\nu_{J-})\right] - iu \ln\left[(1 - \nu_{J+})(1 + \nu_{J-})\right]. \tag{A.72}$$

We next consider the characteristic exponent of the convexity-adjusted jump component under the measure $\mathbb{Q}_{n-1}$, i.e., for $j = n-2$. We obtain

$$\psi_J^{\mathbb{Q}_{n-1}}(u) \approx \int_{\mathbb{R}_0} \left[1 - e^{iux}\right] \left[1 + A_{n-1}\left[e^x - 1\right]\right) \pi_J^{\mathbb{Q}_n} dx - k_J(1) \tag{A.73}$$

$$\begin{aligned} &= [1 - A_{n-1}] \ln\left[(1 - iu\nu_{J+})(1 + iu\nu_{J-})\right] + A_{n-1} \ln\left[(1 - iu\nu_{J+}^{n-2})(1 + iu\nu_{J-}^{n-2})\right] \\ &- iu\left[[1 - A_{n-1}] \ln\left[(1 - \nu_{J+})(1 + \nu_{J-})\right] + A_{n-1} \ln\left[(1 - \nu_{J+}^{n-2})(1 + \nu_{J-}^{n-2})\right]\right], \end{aligned} \tag{A.74}$$

where

$$A_{n-1} = \frac{\delta L(0, T_{n-1})}{\delta L(0, T_{n-1}) + 1}, \quad \nu_{J+}^j = \frac{\nu_{J+}}{1 - (n-1-j)\nu_{J+}}, \quad \nu_{J-}^j = \frac{\nu_{J-}}{1 + (n-1-j)\nu_{J-}}. \tag{A.75}$$

In the same manner, we can recursively proceed to calculate the characteristic exponents of the convexity-adjusted jump component related to the entire family of forward measures corresponding to the other payment dates $T_1, \ldots, T_{n-2}$ in closed form. It remains to obtain the Laplace transforms of the time changes specified in Subsection III.1.1. We start with the deterministic time change, $\mathcal{T}_{T_j}^B$. We directly obtain

$$\mathbb{E}_0^{\mathbb{M}}\left[\exp\left[-\psi_B^{\mathbb{Q}_{j+1}}(u)\mathcal{T}_{T_j}^B\right]\right] = \exp\left[-\frac{1}{2}\left[iu + u^2\right]\int_0^{T_j} \lambda(s, T_j)^2 ds\right]. \tag{A.76}$$

The integral $\int_0^{T_j} \lambda(s, T_j)^2 ds$ results in a lengthy, but closed form, expression.[28] Next, we take a look at $\mathcal{T}_t^W$ and $\mathcal{T}_t^J$. Due to the presence of the nonzero correlation parameter, we need to proceed in

[28]The explicit formula can be obtained upon request.

two steps. First, for the change between the different forward measures, we can rely on the result presented in Proposition 3. Second, when we switch from the forward measure $\mathbb{Q}_{j+1}$ to $\mathbb{M}$, we have to adjust the activity rate dynamics for the time change $\mathcal{T}_t^W$ as follows:

$$d\widetilde{W}_t^{\mathbb{M}} = = d\widetilde{W}_t^{Q_{j+1}} - \langle d\widetilde{W}_t^{\mathbb{Q}_{j+1}}, iu\sqrt{V_t} dW_t^{\mathbb{Q}_{j+1}} \rangle = d\widetilde{W}_t^{Q_{j+1}} - iu\sqrt{V_t}\rho dt, \tag{A.77}$$

Finally, the time change $\mathcal{T}_t^J$ of the jump component remains the same under $\mathbb{M}$ because its driving Brownian motion is assumed to be independent of the different Lévy processes driving the forward LIBOR directly. Hence, the activity rates under the measure $\mathbb{M}$ follow the dynamics

$$dV_t^W = \left[\kappa_W \theta_W - \kappa_j^{\mathbb{M}} V_t^W\right] dt + \sigma_W \sqrt{V_t^W} d\widetilde{W}_t^{\mathbb{M}} \tag{A.78}$$

and

$$d\nu_t^J = \kappa_J(\theta_J - \nu_t^J)dt + \sigma_J \sqrt{\nu_t^J} dZ_t^{\mathbb{M}}, \tag{A.79}$$

with $\kappa_j^{\mathbb{M}} = \kappa_W - \sum_{k=1}^{n-j-1} \frac{\delta L(0,T_{n-k})}{1+\delta L(0,T_{n-k})} \sigma_W \rho - iu\sigma_W \rho$ for $j = 0, 1, \ldots, n-1$. As the affine structure is retained, the transform

$$\mathbb{E}_t^{\mathbb{M}} \left[ \exp\left[ -\psi_W^{\mathbb{Q}_{j+1}}(u) \mathcal{T}_{T_j}^W - \psi_J^{\mathbb{Q}_{j+1}}(u) \mathcal{T}_{T_j}^J \right] \right] \tag{A.80}$$

has the solution

$$\phi^{\mathbb{Q}_{j+1}}(u) = \exp\left[ -a_W(\tau) - b_W(\tau) V_t - a_J(\tau) - b_J(\tau) \nu_t^J \right], \tag{A.81}$$

with the coefficients given in the proposition. $\psi_W(u)$ remains the same regardless of the forward measure under which we perform the pricing, and so we omit the dependence on the forward measure for notational convenience. ■

## Proof of Proposition 6

For simplicity, we fix $t = 0$. Following the same strategy as for the derivation of the caplet price, the swaption value is

$$PS_0(k, T_j, T_{j+N}) \approx \frac{e^{-z_i k}}{\pi} S_0(T_j, T_{j+N}) \int_0^{\infty} e^{-iz_r k} \frac{\phi_{\widetilde{Y}_{T_j}}(z_r - iz_i - i)}{(iz_r + z_i)(iz_r + z_i + 1)} dz_r, \tag{A.82}$$

where $\phi_{\widetilde{Y}_{T_j}}(z-i)$ denotes the characteristic function of the nonhomogenous Lévy process $\widetilde{Y}_{T_j}$ specified under the co-sliding forward swap measure $\mathbb{Q}^N_{j+1}$. The transform $\phi_{\widetilde{Y}_{T_j}}(u)$ takes the form

$$\begin{aligned}
\phi_{\widetilde{Y}_{T_j}}(u) &= \mathbb{E}_0^{\mathbb{Q}^N_{j+1}}\left[\exp\left(iu\widetilde{Y}_{T_j}\right)\right] && \text{(A.83)}\\
&= \mathbb{E}_0^{\mathbb{Q}^N_{j+1}}\left[\exp\left(iu\left[\ln R^N_j(0) + \int_0^{T_j} \sum_{k=j}^{j+N-1} \tilde{\omega}_k(0) dX_s^{\mathbb{Q}^N_{j+1}} + drift\right]\right)\right] && \text{(A.84)}\\
&= R^N_j(0)^{iu} \mathbb{E}_0^{\mathbb{Q}^N_{j+1}}\left[\exp\left(iu\left[B^{\mathbb{Q}^N_{j+1}}_{\mathcal{T}^B_t} - \frac{1}{2}\mathcal{T}^B_{T_j}\right] + iu\left[W^{\mathbb{Q}^N_{j+1}}_{\mathcal{T}^W_t} - \frac{1}{2}\mathcal{T}^W_{T_j}\right]\right.\right.\\
&\quad + \left.\left. iu\left[J^{\mathbb{Q}^N_{j+1}}_{\mathcal{T}^J_{T_j}} - k_J(1)\mathcal{T}^J_{T_j}\right]\right)\right] && \text{(A.85)}\\
&= R^N_j(0)^{iu} \mathbb{E}_0^{\mathbb{M}}\left[\exp\left(-\psi_B^{\mathbb{Q}^N_{j+1}}(u)\mathcal{T}^B_{T_j} - \psi_W^{\mathbb{Q}^N_{j+1}}(u)\mathcal{T}^W_{T_j} - \psi_J^{\mathbb{Q}^N_{j+1}}(u)\mathcal{T}^J_{T_j}\right)\right]. && \text{(A.86)}
\end{aligned}$$

The new measure $\mathbb{M}$ is defined by the complex valued exponential $\mathbb{Q}^N_{j+1}$-martingale,

$$\begin{aligned}
\frac{d\mathbb{M}}{d\mathbb{Q}^N_{j+1}}_{|\mathcal{F}_{t=0}} &= \exp\left[iu\left[B^{\mathbb{Q}^N_{j+1}}_{\mathcal{T}^B_{T_j}} - \frac{1}{2}\mathcal{T}^B_{T_j}\right] + \psi_B^{\mathbb{Q}^N_{j+1}}(u)\mathcal{T}^B_{T_j} + iu\left[W^{\mathbb{Q}^N_{j+1}}_{\mathcal{T}^W_{T_j}} - \frac{1}{2}\mathcal{T}^W_{T_j}\right] + \psi_W^{\mathbb{Q}^N_{j+1}}(u)\mathcal{T}^W_{T_j}\right.\\
&\quad + \left. iu\left[J^{\mathbb{Q}^N_{j+1}}_{\mathcal{T}^J_{T_j}} - k_J(1)\mathcal{T}^J_{T_j}\right] + \psi_J^{\mathbb{Q}^N_{j+1}}(u)\mathcal{T}^J_{T_j}\right] && \text{(A.87)}
\end{aligned}$$

for $j = 0, \ldots, n-N$. By construction we have $\sum_{k=j}^{j+N-1} \frac{\omega_k(0)L(0,T_k)}{R_j^N(0)} = 1$, and therefore, the weights have no impact on the stochastic time changes to the Brownian motion $W$ and the jump component $J$ because these time changes do not depend on the particular maturity $T_j$. Hence, the time changes now become

$$\mathcal{T}_{T_j}^B = \int_0^{T_j} \left[ \sum_{k=j}^{j+N-1} \tilde{\omega}_k(0)\lambda(s,T_k) \right]^2 ds, \qquad \mathcal{T}_{T_j}^W = \int_0^{T_j} V_s^W ds, \qquad \mathcal{T}_{T_j}^J = \int_0^{T_j} \nu_s^J ds \tag{A.88}$$

The pricing problem can now be split again into calculating the characteristic exponents prior to the time changes and the characteristic exponent of the time changes. The characteristic exponents of the two convexity-adjusted continuous Lévy components prior to the time change remain the same as in the LIBOR model setting. To determine $\psi_J^{\mathbb{Q}_{j+1}^N}(u)$, we first look at the jump intensity under the co-sliding forward measure $\mathbb{Q}_{j+1}^N$. It is related to the jump intensity under the terminal co-sliding forward measure $\mathbb{Q}_{n-N+1}^N$ by

$$\pi_J^{\mathbb{Q}_{j+1}^N} \nu_t^J \approx \prod_{k=1}^{n-N-j} \left[ 1 + \frac{\delta R_{n-N+1-k}^N(0)}{1+\delta R_{n-N+1-k}^N(0)} \left[ e^x - 1 \right] \right] \pi_J^{\mathbb{Q}_{n-N+1}^N} \nu^J, \tag{A.89}$$

where $j = 0, 1 \ldots, n-N$ and

$$\pi_J^{\mathbb{Q}_{n-N+1}^N}[x] = \begin{cases} \lambda e^{-\frac{|x|}{\bar{\nu}_+}} |x|^{-1}, & x > 0 \\ \lambda e^{-\frac{|x|}{\bar{\nu}_-}} |x|^{-1}, & x < 0 \end{cases}, \tag{A.90}$$

with

$$\bar{\nu}_+ = \frac{\nu_+}{1-\varphi_2\nu_+}, \quad \bar{\nu}_- = \frac{\nu_-}{1+\varphi_2\nu_-}. \tag{A.91}$$

We require $\lambda, \bar{\nu}_+, \bar{\nu}_- > 0$. By inspection of Eq. (A.90), we see that the change of measure from the terminal forward LIBOR measure to the co-sliding terminal swap measure is driven by the term $\varphi_2$. Clearly, when $N = 1$, we are back in our LIBOR market setting. Next, as in Eq. (A.69) for the

LIBOR specification, we freeze coefficients and set

$$\frac{\delta R^N_{n-N+1-k}(t)}{1+\delta R^N_{n-N+1-k}(t)} \approx \frac{\delta R^N_{n-N+1-k}(0)}{1+\delta R^N_{n-N+1-k}(0)}, \tag{A.92}$$

to obtain

$$\begin{aligned}
\psi_J^{\mathbb{Q}^N_{j+1}}(u) &= \int_{\mathbb{R}_0}\left[1-e^{iux}\right]\prod_{k=1}^{n-N-j}\left[1+\gamma_J(t,T_{n-N+1-k},T_{n+1-k})\right]\pi_J^{\mathbb{Q}^N_{n-N+1}}dx && \text{(A.93)}\\
&\approx \int_{\mathbb{R}_0}\left[1-e^{iux}\right]\prod_{k=1}^{n-N-j}\left[1+\frac{\delta R^N_{n-N+1-k}(0)}{1+\delta R^N_{n-N+1-k}(0)}\left[e^x-1\right]\right]\pi_J^{\mathbb{Q}^N_{n-N+1}}dx, && \text{(A.94)}
\end{aligned}$$

for $j=0,1\ldots,n-N$. Because $R^N_{n-N}(t)$ is a $\mathbb{Q}^N_{n-N+1}$-martingale, we obtain, for $j=n-N$,

$$\begin{aligned}
\psi_J^{\mathbb{Q}^N_{n-N+1}}(u) &= \int_{\mathbb{R}_0}\left[1-e^{iux}\right]\pi_J^{\mathbb{Q}^N_{j+1}}dx - k_J^{n-N}(1) && \text{(A.95)}\\
&= \ln\left[(1-iu\bar{\nu}_{J+})(1+iu\bar{\nu}_{J-})\right] - iu\ln\left[(1-\bar{\nu}_{J+})(1+\bar{\nu}_{J-})\right]. && \text{(A.96)}
\end{aligned}$$

Eq. (A.95) is similar to the characteristic exponent in Eq. (A.71) for the LIBOR, but with $k_J^{n-N}(s)$ as the cumulant exponent of the Lévy jump component with the adjusted jump parameters given in Eq. (A.91). Next, for a given $N$ we have a family of co-sliding forward swap measures denoted by $\left\{\mathbb{Q}^N_1,\mathbb{Q}^N_2\ldots,\mathbb{Q}^N_{n-N+1}\right\}$. To find the characteristic exponent of the convexity-adjusted jump component $\psi_J^{\mathbb{Q}^N_{n-N}}(u)$ related to the swap measure $\mathbb{Q}^N_{n-N}$ prior to the time change, we have to adjust the (co-sliding terminal) Lévy density exactly as we did in the LMM. Working back recursively, and because $R^N_{n-N-1}(t)$ is a $\mathbb{Q}^N_{n-N}$-martingale, we get, for $j=n-N-1$,

$$\begin{aligned}
\psi_J^{\mathbb{Q}^N_{n-N}}(u) &\approx \int_{\mathbb{R}_0}\left[1-e^{iux}\right]\left[1+A_{n-N}\left[e^x-1\right]\right)\pi_J^{\mathbb{Q}^n_{n-N+1}}dx - k_J^{n-N-1}(1) && \text{(A.97)}\\
&= \left[1-A_{n-N}\right]\ln\left[(1-iu\bar{\nu}_{J+})(1+iu\bar{\nu}_{J-})\right] + A_{n-N}\ln\left[(1-iu\bar{\nu}_{J+}^{n-N-1})(1+iu\bar{\nu}_{J-}^{n-N-1})\right]\\
&- iu\left[\left[1-A_{n-N}\right]\ln\left[(1-\bar{\nu}_{J+})(1+\bar{\nu}_{J-})\right] + A_{n-N}\ln\left[(1-\bar{\nu}_{J+}^{n-N-1})(1+\bar{\nu}_{J-}^{n-N-1})\right]\right],
\end{aligned}$$
$$\text{(A.98)}$$

where

$$A_{n-N} = \frac{\delta R^N_{n-N}(0)}{1+\delta R^N_{n-N}(0)}, \quad \bar{\nu}^j_{J^+} = \frac{\bar{\nu}_{J^+}}{1-(n-N-j)\bar{\nu}_{J^+}}, \quad \bar{\nu}^j_{J^-} = \frac{\bar{\nu}_{J^-}}{1+(n-N-j)\bar{\nu}_{J^-}}. \tag{A.99}$$

Following the same argument, we can calculate the characteristic exponents of the convexity-adjusted jump component related to the other co-sliding swap measures $\left\{\mathbb{Q}^N_1, \mathbb{Q}^N_2, \ldots, \mathbb{Q}^N_{n-N-1}\right\}$ in closed form.

As in to the LIBOR market setting, we can show that the activity rate dynamics under the complex measure $\mathbb{M}$ obeys the following form,

$$dV^W_t = \left[\kappa_W \theta_W - \kappa^{\mathbb{M}}_j V^W_t\right] dt + \sigma_W \sqrt{V^W_t} d\widetilde{W}^{\mathbb{M}}_t \tag{A.100}$$

and

$$d\nu^J_t = \kappa_J(\theta_J - \nu^J_t)dt + \sigma_J \sqrt{\nu^J_t} dZ^{\mathbb{M}}_t, \tag{A.101}$$

with $\kappa^{\mathbb{M}}_j = \kappa_W - \varphi_2 \sigma_W \rho - \sum_{k=1}^{n-N-j} \frac{\delta R^N_{n-N+1-k}(0)}{1+\delta R^N_{n-N+1-k}(0)} \sigma_W \rho - iu\sigma_W \rho$ for $j = 0, 1, \ldots, n-N$.[29] Because the time change $\mathcal{T}^B_t$ is deterministic, we obtain

$$\mathbb{E}^{\mathbb{M}}_0 \left[\exp\left[-\psi^{\mathbb{Q}^N_{j+1}}_B(u)\mathcal{T}^B_{T_j}\right]\right] = \exp\left[-\frac{1}{2}\left[iu+u^2\right] \int_0^{T_j} \left[\sum_{k=j}^{j+N-1} \tilde{\omega}_k(0)\lambda(s,T_k)\right]^2 ds\right], \tag{A.102}$$

which can be computed in closed form. We are, therefore, left with the Laplace transform

$$\mathbb{E}^{\mathbb{M}}_0 \left[\exp\left[-\psi^{\mathbb{Q}^N_{j+1}}_W(u)\mathcal{T}^W_{T_j} - \psi^{\mathbb{Q}^N_{j+1}}_J(u)\mathcal{T}^J_{T_j}\right]\right]. \tag{A.103}$$

[29] All the change of measure results applied to the co-sliding swap market model should match the formulae presented for the LMM in the case $N = 1$ corresponding to a co-sliding forward swap rate defined on a three months tenor, in which case it equals the definition of the forward LIBOR.

Due to its exponential affine structure, the Laplace transform has a closed form solution:

$$\phi(u) = \exp\left[-a_W(\tau) - b_W(\tau)V_t - a_J(\tau) - b_J(\tau)\nu_t^J\right], \qquad \tau = T_j - t, \tag{A.104}$$

with

$$a_i(\tau) = \frac{\kappa_i\theta_i}{\sigma_i^2}\left[2\ln\left[1 - \frac{\gamma_i - \hat{\kappa}_i}{2\gamma_i}\left[1 - e^{-\gamma_i\tau}\right]\right] + \left[\gamma - \hat{\kappa}_i\right]\tau\right] \tag{A.105}$$

and

$$b_i(\tau) = \frac{2\psi_i(u)\left[1 - e^{-\gamma_i\tau}\right]}{2\gamma_i - (\gamma_i - \hat{\kappa}_i)\left[1 - e^{-\gamma_i\tau}\right]}, \qquad \gamma_i = \sqrt{\hat{\kappa}_i^2 + 2\sigma_i^2\psi_i(u)}, \tag{A.106}$$

for

$$\psi_i(u) = \begin{cases} \psi_W(u) & \text{if } i = W \\ \psi_J^{\mathbb{Q}_{j+1}^N}(u) & \text{if } i = J \end{cases}, \qquad \hat{\kappa}_i = \begin{cases} \bar{\kappa}_j^{\mathbb{M}} & \text{if } i = W \\ \kappa_J & \text{if } i = J \end{cases}. \tag{A.107}$$

The index $j$ denotes the option maturity under consideration. ■

# Appendix B

In this Appendix, we describe how we transformed the market quotes into the option prices that we used for our model estimation.

### Market prices of caps and floors

Given an implied volatility quote $IV(t, T, K)$ at time $t$ for a cap with maturity date $T$, strike rate $K$, and tenor $\delta$, the invoice (dollar) price of the cap with one dollar principal is computed based on the Black (1976) formula,

$$Cap_t(T, K) = \sum_{i=1}^{N} C_t(T_i, K) = \delta \sum_{i=1}^{N} P(t, t + (i+1)\delta) \left[ L(t, t + i\delta) N(d_{1i}) - K N(d_{2i}) \right], \quad \text{(A.108)}$$

where $T = (N + 1)\delta$. Here, $C_t(T, K)$ is the price of the caplet with maturity $T - t$ and strike $K$, $P(t, T)$ denotes the zero-coupon bond with time to maturity $T - t$, and $L(t, T)$ denotes the forward LIBOR at time $t$ for $[T, T + \delta]$. Also, $N(\cdot)$ denotes the cumulative normal distribution function, and $d_{1i}$ and $d_{2i}$ are defined by,

$$d_{1i} = \frac{\ln(L(t, t + i\delta)/K) + IV^2(i\delta)/2}{IV\sqrt{i\delta}}, \quad d_{2i} = d_{1i} - IV\sqrt{i\delta}. \quad \text{(A.109)}$$

Caplets are paid in arrears. The payoff settled at time $T$ is to be paid one tenor later at $T + \delta$. For US dollar caps and floors, the payment interval is three months, i.e., $\delta = 1/4$. From the Black-implied cap prices, the price of a floor contract can be computed by the parity,

$$Cap_t(T,K) - Floor_t(T,K) \quad = \quad Payer\ Swap_t(T,K), \tag{A.110}$$

which implies that

$$Floor_t(T,K) \quad = \quad Cap_t(T,K) + \sum_{i=1}^{N} \delta P(t, t+(i+1)\delta)(K - L(t, t+i\delta)), \tag{A.111}$$

where the second term on the right-hand side is simply the present value of a (receiver) swap contract. For the estimation of our model, we normalize the option prices by their vega, i.e., their sensitivity to implied volatility ($IV$) changes. For caps (and floors), the vega is given by,

$$\mathcal{V}_{cap/floor} \quad \equiv \quad \frac{\partial Cap_t(T,K)}{\partial IV} = \frac{\partial Floor_t(T,K)}{\partial IV} = \delta \sum_{i=1}^{N} P(t, t+(i+1)\delta) L(t, t+i\delta) N'(d_{1i}) \sqrt{i\delta}, \tag{A.112}$$

where $N'(\cdot)$ is the density of the standard normal distribution.

### Market prices of payer and receiver swaptions

Given an ATM implied swaption volatility quote $IV(t, T_j, T_{j+N})$ at time $t$ for an option expiry at time $T_j$ and a swap tenor equal to $T_N - T_j$ and tenor $\delta$, the invoice (dollar) price of a payer swaption can be computed by relying on the Black (1976) formula for swaptions. In particular, the price for a $T_j \times (T_{j+N} - T_j)$ payer swaption at time $t$ becomes,

$$Payer\ Swaption_t(T_j, T_{j+N}) \quad = \quad S_t(T_j, T_{j+N}) \left[ R(t; T_j, T_{j+N}) N(d_1) - K N(d_2) \right], \tag{A.113}$$

where

$$d_1 = \frac{\ln(R(t;T_j,T_{j+N})/K) + \frac{1}{2}IV(t,T_j,T_{j+N})^2(T_j - t)}{IV(t,T_j,T_{j+N})\sqrt{T_j - t}}, \quad d_2 = d_1 - IV(t,T_j,T_{j+N})\sqrt{T_j - t}. \tag{A.114}$$

Furthermore, $S_t(T_j,T_{j+N}) = \sum_{k=j+1}^{j+N} \delta P(t,T_k)$ and the underlying forward swap rate can be computed by $R(t;T_j,T_{j+N}) = \frac{P(t,T_j)-P(t,T_{j+N})}{S(t,T_j,T_{j+N})}$. From the payer swaption prices, one can apply the put-call parity to obtain receiver swaption prices. Similarly to caps and floors, the ATM swaption strike is defined to be the value of $K$, which makes the payer swaption and receiver swaption prices equal to each other. From this parity relation, it follows that the ATM strike rate is simply the forward starting par swap rate given by $R(t;T_j,T_{j+N})$. For payer and receiver swaptions, the vega is given by,

$$\mathcal{V}_{Swaption} \equiv \frac{\partial Swaption_t(T_j,T_{j+N})}{\partial IV} \quad = \quad S_t(T_j,T_{j+N})R(t;T_j,T_{j+N})N'(d_1)\sqrt{T_j - t}. \tag{A.115}$$

As for caps, we normalize the swaption prices by their Black (1976) vega.

# Chapter 2

## ***Strategic Investment and Optimal Portfolio Choice under Incomplete Markets***

*by Jacob Strømberg*

# Strategic Investment and Optimal Portfolio Choice under Incomplete Markets*

Jacob Strømberg§

Abstract

I analyze the implications of technological innovation and non-diversifiable risk on entrepreneurial entry and optimal portfolio choice. In a real option model where two risk-averse individuals strategically decide over technology adoption, I show that the impact of non-diversifiable risk on the option timing decision is ambiguous and depends on the arrival intensity of the future technology. Consequently, non-diversifiable risk may accelerate or delay the optimal investment decision compared to complete markets. Moreover, strategic considerations regarding technology adoption are shown to play a central role for the optimal portfolio choice in the presence of non-diversifiable risk.

JEL-Classification: G11; G31; E2;
Keywords: Real Options; Incomplete Markets; Technology Adoption; Optimal Portfolio Choice;

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## I. Introduction

Strategic considerations regarding technology adoption, future innovations and non-diversifiable risk in business projects are essential factors affecting individuals' decision to become entrepreneurs. I study the implications of these factors on entrepreneurial entry and optimal portfolio choice in a continuous-time real option model.[1]

Evidence indicates that one of the most profound reasons as to whether people decide to pursue entrepreneurship is (related to) the uncertain income one receives as being an entrepreneur.[2] Not only is entrepreneurship risky due to the variable and uncertain income stream, but since ownership in the business is often substantial (Hall and Woodward (2010), Gentry and Hubbard (2004) and Moskowitz and Vissing-Jorgensen (2002)) the entrepreneur is also exposed to any non-diversifiable income risk. Presence of non-diversifiable risk implies that the market becomes incomplete. Therefore, when modeling entrepreneurial investment behavior, it seems particularly relevant to operate under incomplete markets.

Entrepreneurship has long been regarded as a driving factor of technological change because of private business owners' willingness to adopt and exploit new innovations (e.g., Romer (1990) and Quadrini (2009)). In order to study entrepreneurial investment behavior from a normative perspective another central aspect to incorporate is technological innovation. Moreover, Huisman and Kort (2004) show in complete markets that strategic aspects regarding technology adoption have important implications on both the valuation and timing of investment decisions to enter into markets where technological innovation is important.

[1]The decision by individuals to become entrepreneurs can be viewed as real investments. Due to the fundamental impact real investments have on the economy, the real option approach has received considerable attention in the literature starting with prominent contributions by Brennan and Schwartz (1985) and McDonald and Siegel (1986). A comprehensive overview of classical but also more recent real option contributions can be found in Dixit and Pindyck (1994) and Schwartz and Trigeorgis (2004).

[2]In May 2010, the European Commission published a report (*Entrepreneurship in the EU and beyond*) about entrepreneurship, see European-Commission (2010). Respondents participating in the analysis were asked a range of questions related to entrepreneurship. Asked about the greatest fears when starting up a business, the uncertainty of not having a regular income was mentioned by 40% of the Europeans (and by nearly 50% of the people in US), as being the most important risk of becoming an entrepreneur, entering in second and first place respectively amongst other reasons such as the possibility of going bankrupt, the risk of losing your property, job insecurity etc.

There is currently little research on optimal investment timing decisions under incomplete markets that takes technological innovation and strategic aspects over technology adoption into account.[3] This paper contributes to fill this gap in the literature. Specifically, I address the following question related to the dynamics of entrepreneurship: *What is the impact of non-diversifiable risk on the optimal investment timing decision compared to complete markets, in the presence of strategic considerations regarding technology adoption?*[4]

To analyze this question, I extend the model in Huisman and Kort (2004) to an incomplete market setting where two risk-averse entrepreneurs each have access to an investment opportunity. Each entrepreneur has to strategically decide when to invest, and whether to adopt an existing technology (technology 1) for production or wait until a more efficient technology (technology 2) may become available for adoption. In addition, to decide on the optimal time to exercise their real investment options, they also make optimal intertemporal portfolio decisions as in Merton (1971). This raises another question, I address in the model: *How is the optimal portfolio choice affected by strategic considerations regarding technology adoption?*

The main results of the model can be summarized as follows: First, I show that the impact of non-diversifiable risk on the entrepreneurs' option timing is ambiguous and depends on the arrival intensity of technology 2. Consequently, presence of non-diversifiable risk may accelerate or delay the optimal investment timing compared to complete markets. This result complements the finding in Miao and Wang (2007), who show that the investment timing decision for a single agent who receives a flow payoff over time should always be delayed in the presence of non-diversifiable risk compared to complete markets.

Second, the model gives rise to empirical implications regarding optimal investment timing for

[3]Concurrently while I was working on this paper, I became aware of a related paper by Bensoussan, Diltz, and Hoe (2010). They analyze real options games in complete and incomplete markets with several decision makers from a pure mathematical perspective. The aim of their paper is to mathematically characterize whether a solution exists to certain variational inequalities, characterize its mathematical properties and determine whether it is unique under different conditions. Thus, the authors make no attempt to solve the model numerically nor do they perform any economic analysis about the implications of non-diversifiable risk. Furthermore, they do not consider technological innovations in their model.

[4]This question was motivated by the anecdotal evidence described in Section II.6.

individuals contemplating to become entrepreneurs. For the set of parameters used in the numerical analysis, when it is optimal for one of the entrepreneurs to invest and become the leader, the predominant prediction is that technology adoption by an under-diversified risk-averse entrepreneur should occur sooner under incomplete markets compared to a well-diversified individual or company. This is consistent with other theoretical work that entrepreneurs tend to promote new innovations (see, e.g., Romer (1990) or Schmitz (1989) by way of imitation by established companies). According to the model, a possible explanation for such behavior (at the micro-level) is driven by optimality concerns and risk-aversion: entrepreneurs may take strategic aspects over uncertain future technological innovations and their exposure to non-diversifiable income risk into account prior to investing.

Third, the model offers new insight into the determinants on optimal portfolio choice for both current and prospective entrepreneurs. In environments with greater technological innovation and higher correlation between the operating net-income and the risky asset, the more the prospective entrepreneur (follower) should reduce the portfolio allocation to the risky asset, *already prior* to exercising the investment option, as it becomes more profitable to invest. This finding suggests that not only current entrepreneurs being exposed to non-diversifiable income risk from managing a business should hold more conservative risky asset allocations (e.g., Heaton and Lucas (2000) and Hongyan and Nofsinger (2009) for empirical evidence) it may also apply to prospective entrepreneurs.

In contrast, the current entrepreneur (leader) should conversely increase the portfolio allocation to the risky asset, in anticipation that the follower optimally exercises her investment option, should the more efficient technology arrive. When the follower decides to exercise her investment option, the leader will experience a reduction in operating income from managing the business and also be less exposed to non-diversifiable income risk which *ceteris paribus*, induces a lower hedging demand. The precise effect depends on the relative profitability of operating in the market alone versus operating with an inferior technology. These findings have practical relevance for optimal portfolio choice for both current and prospective entrepreneurs in environments where technological innovation is important.

A fourth and final novel contribution of the paper, is that I jointly relax three commonly used assumptions in the real option literature: (1) the real investment opportunity is always tradable, i.e., it is always instantaneously available; (2) the real asset payoff can be completely spanned by existing traded financial assets; and (3) the agent is risk-neutral.

## I.1. Relation to the Existing Literature

The current paper contributes to several strands of research: (1) to the growing literature on investment under uncertainty in incomplete markets; (2) to the literature on optimal investment timing in the presence of competition; and (3) to studies on entrepreneurial portfolio choice. In this section, I briefly review related papers in these areas of research.

Most papers in the real option literature traditionally rely on the assumption that either the underlying real asset is directly traded or its risk profile can be spanned by trading existing financial assets. In reality, most real assets are not traded in the capital markets and their risk characteristics may at best be partially spanned by the universe of traded financial assets thus leading to incomplete markets. Recently some theoretical development has been made in the literature to extend real option models to incomplete markets. Miao and Wang (2007) study the optimal consumption and portfolio choice for a single entrepreneur who has a single investment opportunity. Henderson (2007) considers a single entrepreneur who has a single investment opportunity to receive a lump-sum payoff. Managerial investment behavior has been analyzed in Hugonnier and Morellec (2007). Evans, Henderson, and Hobson (2008) study the optimal time to sell an asset in the presence of wealth effects. Chen, Miao, and Wang (2010) combines a real option model under incomplete markets with Leland (1994)'s capital structure model. Bensoussan, Diltz, and Hoe (2010) study the mathematical properties real options under incomplete markets for several decision makers. Wang, Wang, and Yang (2012) study the effects of non-diversifiable risk on the optimal investment and exit decisions of a single entrepreneur in the presence of financing and liquidity constraints.[5]

[5]Empirical papers concerned with entrepreneurship and non-diversifiable risk include, Heaton and Lucas (2000), Moskowitz and Vissing-Jorgensen (2002) and Hall and Woodward (2010).

Common for the above papers (except for Bensoussan, Diltz, and Hoe (2010)), is that they only consider the investment decision of a single entrepreneur. In reality, real investment opportunities can rarely be considered in isolation. A distinct branch of the real option literature considers strategic interactions in various forms.[6] An early prominent contribution is Fudenberg and Tirole (1985), who in a deterministic setting, present a theoretical formalization of games in continuous-time. They study technology adoption for two identical firms and show that preemption should happen at the point where rent equalization occurs between the leader and the follower. Stenbacka and Tombak (1994) extend the model setup in Fudenberg and Tirole (1985) by introducing uncertainty into the length of time from initial adoption of a technology and until successful implementation. Similarly, Hoppe (2000) extends the setting in Fudenberg and Tirole (1985) to consider uncertainty regarding the profitability of adopting a new technology. Recently, Thijssen, Huisman, and Kort (2002) have extended the Fudenberg and Tirole (1985) model to a stochastic setting and in a follow-up paper, Huisman and Kort (2004) extend the work by Thijssen, Huisman, and Kort (2002) to identical risk-neutral firms competing over technology adoption while taking into account future technological innovations. However, a standard assumption in the above mentioned papers is that markets are complete.

The final strand of research to which the current paper is related, regards optimal portfolio choice of entrepreneurs. Theoretical studies such as Merton (1971), Viceira (2001), and Heaton and Lucas (2004) document the importance of non-diversifiable risk for individuals' portfolio choice. The notion is that in the presence of uninsurable risk, individuals will adjust their risky asset holdings to partially hedge against unfavorable movements in this risk factor. Empirical studies on entrepreneurial portfolio choice include Guiso, Jappelli, and Terlizzese (1996), Heaton and Lucas (2000) and Hongyan and Nofsinger (2009).

The paper is organized as follows. Section II presents the economic setting. In Section III, the

[6] In the report (*Entrepreneurship in the EU and beyond*) nearly 50% of the European respondents answered, that they do not like situations in which they have to compete with others. Hence, presence of competition may not only generate "affordable" variations in the investment behavior but may in some cases even deter some people from engaging in entrepreneurial activity in the first place.

model is solved and Section IV presents a numerical analysis. Finally, Section V concludes.

## II. The Economic Setting

### II.1. Assumptions on the Entrepreneurs and Technology Adoption

Two identical infinitely lived risk-averse entrepreneurs each have a single irreversible investment opportunity to enter a market which gives them an uncertain flow payoff over time. The entrepreneurs have to strategically decide when to invest and whether to adopt the existing technology (technology 1) for production or whether to wait until a more efficient technology (technology 2) may become available for adoption. Each entrepreneur can only invest once, i.e., it is not possible for the entrepreneurs to upgrade from technology 1 to technology 2. As in Huisman and Kort (2004) the innovation process is assumed to be exogenous. The arrival of technology 2 is modeled according to a Poisson process with constant parameter $\lambda > 0$. Thus, at the random time $T$, which follows an exponential distribution with mean $1/\lambda$, a more efficient technology will be available.[7]

### II.2. Assumptions on the Investment Project (Non-traded Asset)

Let the index $k \in \{i, j\}$ represent each of the entrepreneurs and let $N_k \in \{0, 1, 2\}$ denote whether entrepreneur $k$ has adopted technology 1 ($N_k = 1$), technology 2 ($N_k = 2$) or whether she has not yet entered the market at all ($N_k = 0$). The entrepreneurs can invest by paying a fixed investment cost $I > 0$ equal for both entrepreneurs. After investing, entrepreneur $i$ receives operating net-income from managing the business given by $D_{N_i N_j} Y_t$, where $D_{N_i N_j}$ is a factor representing the technologies adopted. The dynamics of $Y$ equals,

$$\text{(Project value)} \qquad dY_t = \alpha_y dt + \sigma_y \rho dB_t + \sigma_y \sqrt{1-\rho^2} d\tilde{B}_t \qquad Y_0 = y \tag{1}$$

[7]Modeling the arrival of a new technology in this way, helps to ensure that the model setup remains in a time-homogenous setting which facilitates analytical tractability.

where $\{B_t\}_{t\geq 0}$ and $\{\tilde{B}_t\}_{t\geq 0}$ are independent standard Brownian motions. The parameter $\alpha_y$ denotes the expected drift and $\sigma_y$ the volatility of the project value. The parameter $\rho$ denotes the correlation between the project value and the risky asset (see Section II.3). The risks inherent in the investment project cannot be completely hedged by trading in the financial market, i.e., $|\rho| < 1$. Positive values of the process $\{Y_t\}_{t\geq 0}$ denote operating profits whereas negative values denote operating losses.

I follow Huisman and Kort (2004) and specify the following assumptions on the "income factors" representing the technology adopted by the entrepreneurs,

$$
\begin{array}{ccccc}
D_{20} & > & D_{21} & > & D_{22} \\
\vee & & \vee & & \vee \\
D_{10} & > & D_{11} & > & D_{12}
\end{array}
$$

The entrepreneurs receive zero operating income if they have not invested, i.e., $D_{0N_k} = 0$ for $N_k \in \{0, 1, 2\}$. The structure on the factors, $D_{N_iN_j}$, implies that technology 2 is superior to technology 1, and adopting either technology before the competitor yields a higher flow payoff compared to the situation where the competitor has adopted the same technology.

## II.3. Assumptions on the Financial Market

In addition to investing in the business project, the entrepreneurs have access to a risk-free bond, $P$, paying a constant interest rate $r > 0$, and a risky asset, $S$, which is traded in the financial markets. The dynamics of the risky asset is assumed to follow a geometric Brownian motion. The financial investment universe is defined as follows,

$$
\begin{array}{lll}
\text{(Risky asset)} & dS_t = \mu_S S_t dt + \sigma_S S_t dB_t & S_0 > 0 \qquad (2) \\
\text{(Bank account)} & dP_t = rP_t dt & P_0 > 0 \qquad (3)
\end{array}
$$

where $\mu_S$ and $\sigma_S$ are constant parameters respectively denoting the expected rate of growth and volatility of the risky asset.

## II.4. Assumptions on the Valuation Methodology

The implication of working under incomplete markets is that the usual valuation techniques from no-arbitrage option pricing theory (e.g., Black and Scholes (1973)) cannot be applied since a standard replication argument no longer applies.[8] Furthermore, under incomplete markets there exists no unique measure under which we can evaluate real asset investment opportunities to obtain a unique price. As a consequence, we have to resort to other pricing methods suitable under incomplete markets.

One such method (which will be adopted in this paper) is obtained by making explicit assumptions about the entrepreneurs' attitude toward risk via a utility function. When pricing is embedded in a utility maximization setting, a unique (martingale) measure emerges, which can then be used for valuation of non-traded investment opportunities using dynamic programming techniques (e.g., Henderson (2009) or Birge and Linetsky (2007)). This is termed *utility indifference pricing* and gives rise to so-called *certainty-equivalent* valuation.[9] Because the valuation is performed via utility indifference pricing, the entrepreneurs' preferences toward risk have an impact on the option values characterizing the investment opportunities.[10]

## II.5. Assumptions on the Entrepreneurs' Decision Problem

The entrepreneurs are endowed with an exponential utility function over consumption $c$, specified by $u(c) = -\frac{1}{\gamma}e^{-\gamma c}$, where the parameter $\gamma > 0$ denotes the coefficient of absolute risk-aversion. The

[8] That is, it is not possible to construct a replicating portfolio from existing financial assets which match the payoff and risk profile of the investment project perfectly.

[9] The certainty-equivalent valuation is the amount of wealth ($W_{ce}$), which makes the entrepreneur indifferent in utility terms between, (a) having the investment option to adopt technology 1 or 2 and receive an uncertain income stream, and (b) giving up the investment option and instead receive the amount ($W_{ce}$).

[10] This is different from complete markets where the option values are independent of risk-preferences.

choice of utility function is chosen for analytical tractability and the two entrepreneurs are assumed to have the same level of risk-aversion.[11] The objective for each of the entrepreneurs is to maximize the expected discounted utility from consumption over an infinite investment horizon,

$$\mathbb{E}\left[\int_0^\infty e^{-\beta s} u(c_s) ds\right] \tag{4}$$

where $\beta > 0$ denotes the entrepreneurs subjective discount rate. The entrepreneurs optimize the objective with respect to investment and consumption strategy, $\pi$ and $c$, and strategically decide on the optimal time to undertake the investment opportunity.

### II.6. Anecdotal Evidence

It is instructive to think of the modeling setup having the following real-world example in mind: In 2007, the entrepreneur Shai Agassi founded the California-based company *Better Place* - an electric vehicles service provider with a vision to make zero-emission cars. In the Harvard Business Review, May 2009, Shai Agassi talks about technology adoption:

> "*Every night I went to Wikipedia, picked a term like "ethanol" or "natural gas," and studied for hours. Eventually I wrote a white paper proposing a plan that relies on existing technology: cars that run on lithium-ion batteries recharged by renewable energy.*" (Akresh-Gonzales (2009))

We can think of electric-driven vehicles as the existing technology (technology 1). Technological innovation plays a fundamental role in the market for zero-emission vehicles:

> "*The market outlook for electric vehicles seems bright [...] Yet the future of electric vehicles is far from assured. [...] Will other technologies - such as hybrid cars or vehicles*

[11] Assuming different levels of risk-aversion for the entrepreneurs would imply that the one with the lowest risk-aversion would always invest before its competitor. The same argument holds for differentiated investment costs, (e.g., Pawlina and Kort (2006)) Without allowing for, e.g., incomplete information as in Lambrecht and Perraudin (2003) assuming different levels of risk-aversion and/or investment costs appears to be less interesting since the order of investment is given *a priori*.

*powered by natural gas, ethanol, or hydrogen - emerge and win the competition against electric cars?*" (Graham and Messer (2011))

Upon arrival of a more efficient technology (technology 2) other entrepreneurs are likely to invest and compete with the technology adopted by Shai Agassi.

## III. Derivation of the Value Functions

In this section, the model is solved using stochastic dynamic programming. Section III.1 derives the project value payoff the entrepreneurs obtain from exercising their investment options. In Section III.2, the (option) value functions are derived in the case where technology 2 is available. In Section III.3 the (option) value functions are derived before technology 2 has arrived by relying on the expressions derived in Section III.2.

### III.1. The Project Value

**Proposition 1.** *The project value associated with income factor, $D_{N_iN_j}$, in the presence of non-diversifiable risk, is given by the function,*

$$f(y; D_{N_iN_j}) = \frac{D_{N_iN_j}}{r}y + \frac{(\alpha_y - \rho\sigma_y\eta)D_{N_iN_j}}{r^2} - \frac{\gamma\sigma_y^2(1-\rho^2)D_{N_iN_j}^2}{2r^2} \tag{5}$$

To provide intuition for the expression in Proposition 1, consider the situation where the risk-aversion parameter goes to zero ($\gamma \to 0$) and the correlation between the operating net-income and the risky asset goes to one ($\rho \to 1$). Then we approach the traditional complete market setting with risk-neutral entrepreneurs. Under complete markets the risk-neutral dynamics of the operating net-income equals $dY_t = \left(\alpha_y - \sigma_y\eta\right)dt + \sigma_y dB_t^{\mathbb{Q}}$ where $\eta = \frac{\mu_S - r}{\sigma_S}$ denotes the market price of risk.[12]

[12]Under the risk-neutral measure, $\mathbb{Q}$, the risky asset commands a drift equal to the risk-free rate. Then by relying on Girsanov's theorem (e.g., Bjork (2004)) we have that $dB_t = h_t dt + dB_t^{\mathbb{Q}}$ where $B^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion and the Girsanov kernel $h_t$, equals $h_t = -\eta$ where $\eta = \frac{\mu_S - r}{\sigma_S}$ denotes the market price of risk and the risk-neutral dynamics of the operating net-income under complete markets follows.

Consequently, the project value can be computed as the expected present value under the risk-neutral measure, $\mathbb{Q}$, of the flow payoff $D_{N_iN_j}Y_t$ discounted by the risk-free rate,

$$f(y; D_{N_iN_j}) \quad = \quad \mathbb{E}^{\mathbb{Q}}_{Y_0=y}\left(\int_0^{\infty} e^{-rt} D_{N_iN_j} Y_t dt\right) = \frac{D_{N_iN_j}}{r} y + \frac{\left(\alpha_y - \sigma_y \eta\right) D_{N_iN_j}}{r^2}. \tag{6}$$

The project value under complete markets corresponds to the two first terms in the project value under incomplete markets. The last term in the project value under incomplete markets (see Proposition 1), accounts for the risk attitude of the entrepreneurs and the non-diversifiable income risk which lower the valuation relative to complete markets. In contrast to the valuation under incomplete markets, the entrepreneurs' risk-preferences do not appear in the project value under complete markets.

## III.2. The Situation when Technology 2 is Available

In this section, we consider the situation where technology 2 is available. There are three possible scenarios: First, a scenario where none of the entrepreneurs have invested before time $T$ (Section III.2.1); Second, a scenario where one entrepreneur has invested before time $T$ and has become the leader (Section III.2.2); Third a scenario where both entrepreneurs have invested before time $T$ (Section III.2.3). Below the (option) value functions and investment thresholds (where relevant) for each of the scenarios are derived.

### III.2.1. None of the Entrepreneurs have Invested before Time $T$

Because $t \geq T$ both entrepreneurs will adopt technology 2 since it is assumed to be superior to technology 1. Thus, we can view this situation as a single entrepreneur who has to decide when to invest in technology 2. Having determined the project value in Section III.1, we can now determine the associated (option) value function. Proposition 2 gives the option value and the optimal investment threshold for the entrepreneurs.

**Proposition 2.** *The value function for joint investment in technology 2 is given by,*

$$J_{22;AI}(y) = \begin{cases} g(y) & \text{if} \quad y \in (-\infty, \bar{y}_{22}) \\ f(y; D_{22}) - I & \text{if} \quad y \in (\bar{y}_{22}, +\infty) \end{cases}$$

*after time $T$ and where $\bar{y}_{22}$ denotes the optimal investment threshold. The function $g(y)$ satisfies the following non-linear ODE,*

$$rg(y) = \frac{\sigma_y^2}{2} g''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) g'(y)^2 + (\alpha_y - \rho\sigma_y\eta) g'(y) \tag{7}$$

*on $y \in (-\infty, \bar{y}_{22})$ subject to $\lim_{y\to-\infty} g(y) = 0$ and the value-matching and smooth-pasting conditions, $g(\bar{y}_{22}) = f(\bar{y}_{22}; D_{22}) - I$ and $g'(\bar{y}_{22}) = f'(\bar{y}_{22}; D_{22})$ where the function $f(\cdot)$ is given in Proposition 1. The optimal portfolio policy of the entrepreneurs is given in Appendix A.*

The function $g(y)$ represents the option value of joint investment using technology 2 in the continuation region and $f(y; D_{22}) - I$ denotes the intrinsic value of the option in the stopping region. As the operating net-income approaches minus infinity the option value becomes worthless. There exists no closed-form expression for the function $g(y)$ which satisfies the boundary conditions and the solution has to be found numerically.[13] When the operating net-income hits the endogenously determined threshold $\bar{y}_{22}$ both entrepreneurs exercise their investment option, pay the investment costs $I$ and receive the project value $f(y; D_{22})$.

### III.2.2. One Entrepreneur has Invested before Time $T$

Next, we consider the case where one entrepreneur is assumed to have invested before time, $T$, in technology 1 (the leader). The entrepreneur who has not yet invested (the follower), faces a situation of a single individual who has to decide when to invest in technology 2. The follower's option value

[13] Throughout the paper, projection methods using Chebyshev collocation have been applied to solve the differential equations numerically. This approach has proven to be far superior to, e.g., conventional finite difference methods when trying to numerically approximate the solution to non-linear differential equations subject to a free-boundary (see, e.g., Judd (1992) and Dangl and Wirl (2004)).

function and investment threshold are given in Proposition 3.

**Proposition 3.** *The value function for the follower after time $T$, is given by,*

$$F_{21;AI}(y) = \begin{cases} g(y) & \text{if} \quad y \in (-\infty, \bar{y}_{12}) \\ f(y; D_{21}) - I & \text{if} \quad y \in (\bar{y}_{12}, +\infty) \end{cases}$$

*where $\bar{y}_{12}$ denotes the optimal investment threshold. The function $g(y)$ satisfies the following non-linear ODE,*

$$rg(y) = \frac{\sigma_y^2}{2} g''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) g'(y)^2 + (\alpha_y - \rho\sigma_y\eta) g'(y) \tag{8}$$

*on $y \in (-\infty, \bar{y}_{12})$ subject to $\lim_{y\to-\infty} g(y) = 0$ and the value-matching and smooth-pasting conditions $g(\bar{y}_{12}) = f(\bar{y}_{12}; D_{21}) - I$ and $g'(\bar{y}_{12}) = f'(\bar{y}_{12}; D_{21})$ where the function $f(\cdot)$ is given in Proposition 1. The optimal portfolio policy of the follower is given in Appendix A.*

Having determined the value function $F_{21;AI}(y)$ for the follower we are in a position to determine the value function to the leader, i.e, for the entrepreneur who has invested in technology 1. The result is given in Proposition 4.

**Proposition 4.** *The value function for the leader after time $T$, is given by,*

$$L_{12;AI}(y) = \begin{cases} g(y) & \text{if} \quad y \in (-\infty, \bar{y}_{12}) \\ f(y; D_{12}) & \text{if} \quad y \in (\bar{y}_{12}, +\infty) \end{cases}$$

*where $\bar{y}_{12}$ denotes the follower's optimal investment threshold determined in Proposition 3. The function $g(y)$ satisfies the following non-linear ODE,*

$$rg(y) = \frac{\sigma_y^2}{2} g''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) g'(y)^2 + (\alpha_y - \rho\sigma_y\eta) g'(y) + D_{10} y \tag{9}$$

*on $y \in (-\infty, \bar{y}_{12})$ subject to $\lim\limits_{y\to-\infty} g(y) = f(y; D_{10})$ and the value-matching condition $g(\bar{y}_{12}) = f(\bar{y}_{12}; D_{12})$ where the function $f(\cdot)$ is given in Proposition 1. The optimal portfolio policy of the leader is given* *in Appendix A.*

The value-matching condition captures that the leader faces a reduction in the flow payoff from managing the business when the follower invests in technology 2.[14]

### III.2.3. Both Entrepreneurs have Invested before Time $T$

In this scenario both entrepreneurs are assumed to have invested in technology 1 before time, $T$. The value function is given in Proposition 5.

**Proposition 5.** *The value function for joint investment in technology 1 is given by* $J_{11;AI}(y) = f(y; D_{11})$ *where the function* $f(\cdot)$ *is given in Proposition 1.*

## III.3. The Situation before the Arrival of Technology 2

Next, we consider the situation before technology 2 has arrived. In Section III.3.1 the follower's value function is derived. In Section III.3.2 the leader's value function is derived by relying on the follower's optimal investment behavior. In Section III.3.3 and III.3.4 the value functions for joint investment in technology 2 and 1 are respectively derived.

### III.3.1. The Follower's Value Functions

We identify two scenarios for which we can derive the option value functions for the follower: (1) The follower waits for technology 2 before investing, given the leader has invested in technology 1 (Proposition 6); and (2) The follower considers investing in technology 1, given the leader has invested in technology 1 (Proposition 7).

**Proposition 6.** *The value function for the follower before time $T$ (given the follower waits for technology 2) is given by* $F_{21;BI}(y) = G(y)$ *where the function* $G(y)$ *satisfies the following non-linear*

[14] Note, that to determine the value function for the leader, we do not need to invoke any smooth-pasting condition because it is not a free-boundary value problem. Moreover, since we assume the leader has already invested in technology 1, we do not subtract the investment costs in the value-matching condition.

*ODE,*

$$rG(y) = \frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) + \lambda(F_{21;AI}(y) - G(y)) \qquad (10)$$

*on $y \in (-\infty, \infty)$ where $F_{21;AI}(\cdot)$ is given in Proposition 3, subject to the boundary conditions $\lim_{y\to-\infty} G(y) = 0$ and $\lim_{y\to+\infty} G(y) = Ay + B$. The expressions for $A$ and $B$ and the optimal portfolio policy of the follower are given in Appendix A.*

The non-linear ODE that determines the function $G(y)$ has to be solved numerically.[15] The lower boundary condition, $\lim_{y\to-\infty} G(y) = 0$, captures that the option to invest loses its value, as the operating net-income approaches minus infinity. The upper boundary condition represents the expected net-present value of the flow payoff $D_{21}Y_t$ accruing to the follower from investing at time $T$ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging (see Appendix A for additional details).

Next, we consider the scenario where the follower considers investing in technology 1. According to Huisman and Kort (2004), and in line with intuition, such a situation would only be an attractive option for the follower for sufficiently low values of $\lambda$.

**Proposition 7.** *The value function for the follower before time $T$, is given by,*

$$F_{11;BI}(y) = \begin{cases} G(y) & \text{if} \quad y \in (-\infty, \bar{y}_{11}) \\ f(y; D_{11}) - I & \text{if} \quad y \in (\bar{y}_{11}, +\infty) \end{cases}$$

*where $\bar{y}_{11}$ denotes the optimal investment threshold for the follower to invest in technology 1. The*

[15] Note, that in order to solve for $G(y)$ under complete markets, one would rely on the continuity and differentiability conditions around the threshold $\bar{y}_{12}$ (which is known under both complete and incomplete markets) to determine the relevant parameters. However, under incomplete markets, we do not know the value taken by the function $G(y)$ in this point and therefore we cannot utilize the information to determine the function $G(y)$ numerically since we have no closed-form expression to guide us. Thus, we have to solve the differential equations subject to the boundary conditions given in the proposition.

*function $G(y)$ satisfies the following non-linear ODE,*

$$rG(y) = \frac{\sigma_y^2}{2} G''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) + \lambda(F_{21;AI}(y) - G(y)) \tag{11}$$

*on $y \in (-\infty, \bar{y}_{11})$ where $F_{21;AI}(\cdot)$ is given in Proposition 3, subject to $\lim_{y\to-\infty} G(y) = 0$ and the value-matching and smooth-pasting conditions, $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11}) - I$ and $G'(\bar{y}_{11}) = f'(\bar{y}_{11}; D_{11})$ where the function $f(\cdot)$ is given in Proposition 1. The optimal portfolio policy of the follower is given in Appendix A.*

Note, that we subtract the investment costs in the value-matching condition. This represents the case where the operating net-income process reaches (or is above) $\bar{y}_{11}$ and the follower invests in technology 1 and pays the investment costs.

### III.3.2. The Leader's Value Functions

Next, we derive the value functions for the leader under the assumption of immediate investment in technology 1.[16] We identify two scenarios: (1) The follower waits for technology 2 (Proposition 8); and (2) The follower considers investing in technology 1 (Proposition 9).

**Proposition 8.** *The value function for the leader before time $T$ (given the follower waits for technology 2), is given by $L_{12;BI}(y) = G(y) - I$ where the function $G(y)$ satisfies the following non-linear ODE,*

$$rG(y) = \frac{\sigma_y^2}{2} G''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + \left(\alpha_y - \rho\sigma_y\eta\right) G'(y) + D_{10}y + \lambda\left(L_{12;AI}(y) - G(y)\right) \tag{12}$$

*on $y \in (-\infty, \infty)$ where $L_{12;AI}(\cdot)$ is given in Proposition 4, subject to the boundary conditions $\lim\limits_{y\to-\infty} G(y) = f(y; D_{10})$ and $\lim_{y\to\infty} G(y) = Ay + B$. The expression for $A$ and $B$ and the optimal portfolio policy of the leader are given in Appendix A.*

[16] Thus we simply subtract the investment costs at the end, since we assume immediate investment by the leader. They are not part of the problem as it was the case for the follower since we are not explicitly concerned with the exact entry point by the leader at this stage. This is conventional to do when considering real options and competition (see, e.g., Pawlina and Kort (2006)).

The lower boundary condition captures that the leader is exposed to any downside movements in the operating net-income. As the operating net-income approaches minus infinity, it is not optimal for the follower to invest regardless of the technology available and therefore the leader's value function converges to the project value $f(y; D_{10})$. The upper boundary condition captures the expected present value of receiving the flow payoff $D_{10}Y_t$ (from immediate investment in technology 1) up to time $T$ and from that point onwards the flow payoff $D_{12}Y_t$ in perpetuity adjusted for risk-aversion and incomplete hedging.

**Proposition 9.** *The value function for the leader before time $T$, is given by,*

$$L_{11;BI}(y) = \begin{cases} G(y) - I & \text{if} \quad y \in (-\infty, \bar{y}_{11}) \\ f(y; D_{11}) - I & \text{if} \quad y \in (\bar{y}_{11}, +\infty) \end{cases}$$

*where $\bar{y}_{11}$ denotes the follower's optimal investment threshold for investment in technology 1 determined in Proposition 7. The function $G(y)$ satisfies the following non-linear ODE,*

$$rG(y) = \frac{\sigma_y^2}{2} G''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) G'(y)^2 + \left(\alpha_y - \rho\sigma_y\eta\right) G'(y) + D_{10}y + \lambda \left(L_{12;AI}(y) - G(y)\right) \quad (13)$$

*on $y \in (-\infty, \bar{y}_{11})$ where $L_{12;AI}(\cdot)$ is given in Proposition 4, subject to $\lim_{y \to -\infty} G(y) = f(y; D_{10})$ and the value-matching condition $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11})$ where the function $f(\cdot)$ is given in Proposition 1. The optimal portfolio policy of the leader is given in Appendix A.*

The upper boundary condition, $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11})$, captures the reduction in the flow payoff the leader incurs, at the moment when the follower invests in technology 1.

### III.3.3. Waiting to Invest until Technology 2 Arrives

Next, we derive the option value function when both entrepreneurs wait for technology 2.

**Proposition 10.** *The value function for joint waiting until technology 2 arrives, before time $T$,* *is given by* $J_{22;BI}(y) = G(y)$ *where The function* $G(y)$ *satisfies the following non-linear ODE,*

$$rG(y) = \frac{\sigma_y^2}{2} G''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) + \lambda(J_{22;AI}(y) - G(y)) \tag{14}$$

*on* $y \in (-\infty, \infty)$ *where the function* $J_{22;AI}(y)$ *is given in Proposition 2, subject to the boundary conditions* $\lim_{y\to-\infty} G(y) = 0$ *and* $\lim_{y\to+\infty} G(y) = Ay + B$. *The expressions for* $A$ *and* $B$ *and the optimal portfolio policy of the entrepreneurs are given in Appendix A.*

The upper growth condition in Proposition 10 represents the expected net-present value of the flow payoff $D_{22}Y_t$ accruing to both entrepreneurs from investing at time $T$ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging.

### III.3.4. Joint Investment in Technology 1

Finally, Proposition 11 states the value function for joint investment in technology 1.

**Proposition 11.** *The value function for joint investment in technology 1 before time* $T$ *is given by,* $J_{11;BI}(y) = f(y; D_{11}) - I$ *where the function* $f(\cdot)$ *is given in Proposition 2. The optimal portfolio policy is given in Proposition 5.*

## IV. Numerical Analysis

In this section, a numerical analysis of the model is presented. The differential equations defining the (option) value functions in Section III, allow us to study how non-diversifiable risk and technological innovation affect the optimal strategic investment behavior of the risk-averse entrepreneurs compared to the benchmark setting with risk-neutral agents and complete markets.

This paper extends the complete market model in Huisman and Kort (2004) to incomplete markets. Because closed-form solutions exist under complete markets, Huisman and Kort are able to explicitly characterize the different equilibria that can arise depending on the value of the arrival intensity of technology 2. In the presence of non-diversifiable risk, it is no longer possible to obtain closed-form solutions for the (option) value functions and investment thresholds. However, by restricting the model to risk-neutral entrepreneurs ($\gamma \to 0$) and complete markets ($\rho \to 1$), we can find the thresholds of $\lambda$ which characterize the equilibria that occur with respect to the stochastic process considered in this paper. This will serve as the benchmark to which we compare the model predictions under incomplete markets. We merely state the result here and refer to their paper for a complete treatment: A preemptive equilibrium occurs for $\lambda \in [0, \frac{rD_{10}}{D_{21}-D_{12}})$ and for $\lambda < \frac{rD_{11}}{D_{21}-D_{11}}$, the follower considers investing in technology 1. That is, for values of $\lambda$ greater than $\frac{rD_{11}}{D_{21}-D_{11}}$ the threshold $\bar{y}_{11}$ does not exist. For $\lambda \in [\frac{rD_{10}}{D_{21}-D_{12}}, \frac{rD_{10}}{D_{22}-D_{12}})$, an attrition equilibrium arises where the follower is better off than the leader. Finally, for $\lambda \in [\frac{rD_{10}}{D_{22}-D_{12}}, \infty)$, a waiting equilibrium arises, where it is profitable for both entrepreneurs to wait for technology 2 to arrive before investing.[17]

The strategic and optimal investment timing decisions are analyzed in Sections IV.1 to IV.4 for varying levels of technological innovation. Section IV.5 discusses the implications on the optimal portfolio choice of the entrepreneurs. Finally, Section IV.6 provides comparative statics of the optimal investment behavior with respect to certain key parameters.

### IV.1. A Preemption Equilibrium

Consider first the situation where $\lambda < \frac{rD_{10}}{D_{21}-D_{12}} = 0.2667$.[18] Panel A and C in Figure 1 show the (option) value functions for an arrival intensity $\lambda = 0.02$ under complete and incomplete markets respectively. Panel B and D show a similar picture for $\lambda = 0.15$. The panels thus allow for a comparison of how the valuation and optimal investment timing of the follower and the leader changes when we move from complete to incomplete markets. We discuss each of the entrance

[17]Under incomplete markets, it appears difficult to establish a similar general result about the equilibria occurring depending on the value of $\lambda$, since closed-form solutions are absent. However, since the project value, the entrepreneurs obtain after investing, is linear in the operating net-income under both complete and incomplete markets, the upper growth conditions are also linear in the operating net-income under the two market settings. Thus, up to certain parameter combinations which may introduce large non-linear effects in the (option) value functions under incomplete markets, the equilibria defined by the critical values of $\lambda$ under complete markets, can be transferred to the incomplete market setting. This is because the upper growth conditions to a large extent define the critical values of $\lambda$, cf. Huisman and Kort (2004).

[18]To assess the impact of the arrival intensity on the optimal investment timing, $\lambda$ is varying whereas all the other parameters are fixed throughout Figure 1-3.

points in turn.

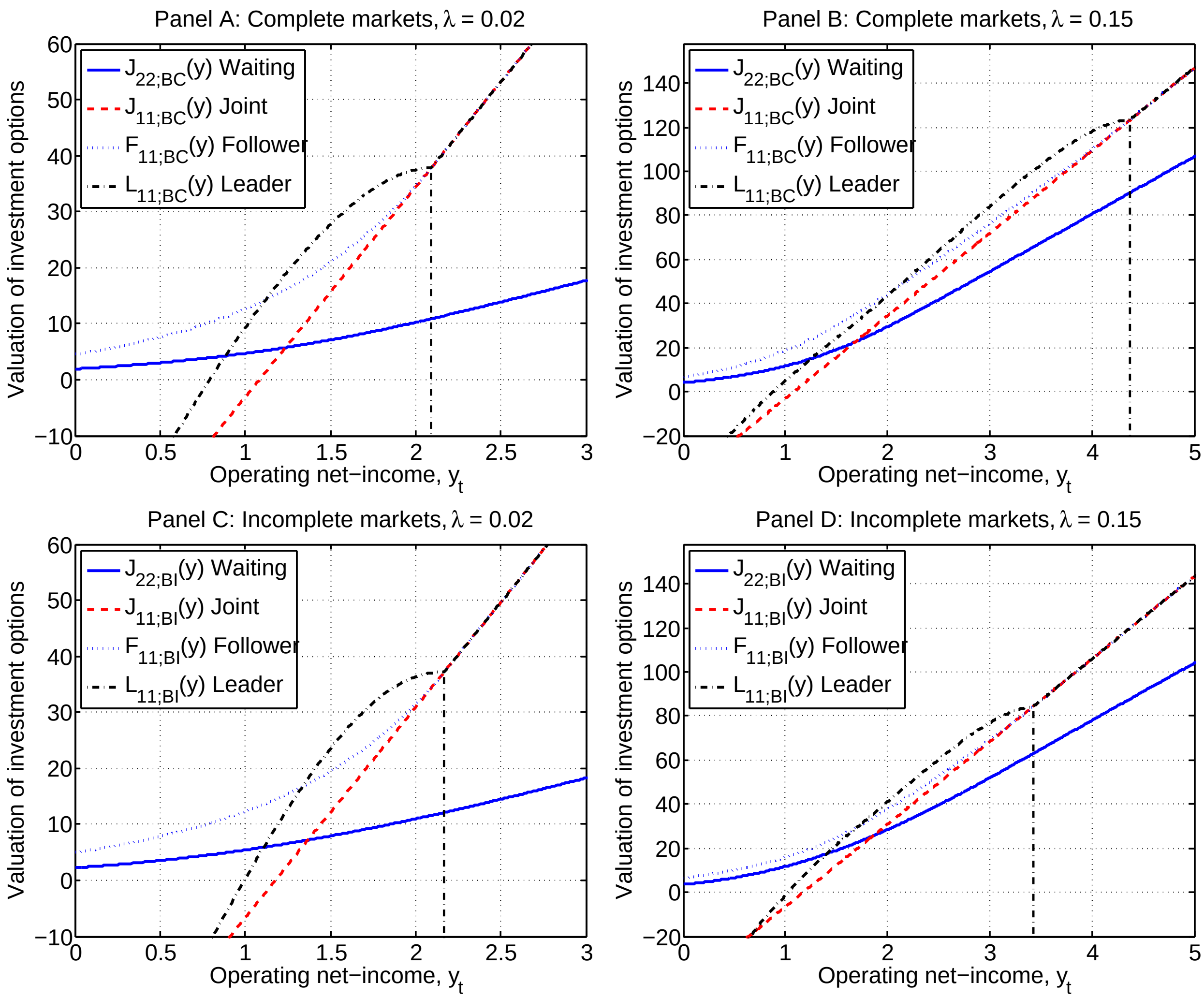


**Figure 1. Option value functions.** Panel A and B show the value functions under complete markets for $\lambda = 0.02$ and $\lambda = 0.15$ and Panel C and D show the value functions under incomplete markets for $\lambda = 0.02$ and $\lambda = 0.15$, respectively. The dash-dotted line denotes the value function for the leader, the dotted line denotes the value function for the follower, the solid line denotes the value function for joint waiting for technology 2 and the dashed line denotes the value function for joint investment in technology 1. The other parameters are specified as follows: project value drift $\alpha_y = 0.05$, project value volatility $\sigma_y = 30\%$, risk-free interest rate $r = 8\%$, investment costs $I = 50$, risky asset drift $\mu_s = 0.10$, risky asset volatility $\sigma_s = 20\%$, risk-aversion $\gamma = 0.25$, correlation between operating net-income and risky asset $\rho = 0.15$. The income factors are, $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

### IV.1.1. The Follower's Investment Threshold

We first examine the follower's optimal investment threshold. The investment threshold $\bar{y}_{11}$ under incomplete markets is determined by trading off the relative changes of the project value $f(y; D_{11})$ and the option value $G(y)$ compared to complete markets (e.g., Miao and Wang (2007)). When the follower invests in technology 1, the value functions for the leader and follower coincide and equals the joint investment value, $f(y; D_{11})$. This happens at the point where the three curves (the leader, the follower and the joint investment curve) collapse into one, as indicated by the vertical dotted line(s) in Figure 1.

Note first, that risk-aversion and incomplete hedging lower both the option value and the project value compared to complete markets. This is intuitive, since the entrepreneurs face non-diversifiable income risk (both before and after option exercise) to which they are risk-averse and that *ceteris-paribus* has a negative impact on the valuations.

From panel A and C in Figure 1 we observe that when technological innovation is sufficiently sparse (e.g., $\lambda = 0.02$), option exercise should be delayed under incomplete markets relative to complete markets. This occurs because risk-aversion and incomplete hedging lower the project value (i.e., the payoff from investing) more relative to the option value.[19]

We also observe that the expected certainty-equivalent gain from arrival of technology 2 has a positive impact on the follower's option value under both market settings. In line with intuition, the more likely technology 2 is to arrive, the more valuable it is for the follower to wait for technology 2, and the larger the operating net-income needs to be, in order to justify investment in technology 1. But in contrast to the situation when technological innovation is sparse (e.g., $\lambda = 0.02$), we observe from Panel B and D that when technological innovation is more concentrated (e.g., $\lambda = 0.15$), investment by the follower should instead be accelerated under incomplete markets compared to

[19] If we let the arrival intensity of technology 2 go to zero ($\lambda \to 0$), we essentially approach the model in Miao and Wang (2007). They show (in a setting with a single risk-averse entrepreneur and without technological innovation), that investment in a project yielding a flow payoff will always be delayed under incomplete markets compared to complete markets. In light of their finding, the prediction of delayed option exercise under incomplete markets for levels of technological innovation close to zero is as expected.

complete markets.

By virtue of risk-aversion and incomplete hedging, the entrepreneurs attach a lower value to the investment opportunity in technology 2 under incomplete markets. Thus, the expected certainty-equivalent gain from arrival of technology 2 has a smaller positive impact on the follower's option value under incomplete markets than under complete markets. This is evident in Figure 1. In particular, the larger the arrival intensity, the smaller is the positive contribution to the option value under incomplete markets relative to complete markets. This can be motivated, by rewriting the expected certainty-equivalent gain from arrival of technology 2 under incomplete markets into a component which resembles the certainty-equivalent gain under complete markets plus an additional term as follows: $\lambda(F_{21;AI}(y) - G(y)) \approx \lambda(F_{21;AC}(y) - G(y)) - \lambda(F_{21;AC}(y) - F_{21;AI}(y))$ where the subscript $I$ denotes "incomplete markets" and $C$ denotes "complete markets". The additional term $\lambda(F_{21;AC}(y) - F_{21;AI}(y))$ is positive, and thus has a negative impact on the option value under incomplete markets, which is increasing in the degree of technological innovation.

Hence, there exists a level of technological innovation, above which, the smaller certainty-equivalent gain under incomplete markets, ends up lowering the option value more relative to the project value because the project value itself is unaffected by the degree of technological innovation. This implies, that the overall *positive effect* on the option value of waiting in the environment with sparse (or without) technological innovation is reversed, thereby leading to accelerated investment in technology 1 by the follower under incomplete markets.

### IV.1.2. The Leader's Investment Threshold

Next, we study the optimal investment threshold for the leader. According to Fudenberg and Tirole (1985) preemption by the leader should occur when there is rent equalization between the leader and the follower (i.e., when the leader and the follower curve intersect each other). We therefore need to examine how the leader and follower value functions are affected as we move from complete to incomplete markets.

When the operating net-income decreases, the leader's value function is lowered more relative to the follower's value function as we move from complete to incomplete markets. Specifically, risk-aversion and non-diversifiable risk reduce the value attained by the lower boundary condition in the leader's valuation problem and that again results in a lower value function under incomplete markets. The follower, on the other hand, has no direct exposure to movements in the operating net-income since she has not exercised her investment option.[20] As a result, the follower's value function converges to zero as the operating net-income decreases regardless of the market setting. From Panel A and C, we observe that when $\lambda = 0.02$, preemption under complete markets occurs at a level of operating net-income, where the leader's value function under incomplete markets is lowered more than the follower's value function. This results in delayed investment under incomplete markets.

In contrast, when $\lambda = 0.15$ preemption should be accelerated under incomplete markets. For increasing operating net-income, risk-aversion and non-diversifiable risk lower the instantaneous expected drift in the follower's value function more than in the leader's value function. That is, the follower's value function exhibits a lower option's *Delta* under incomplete markets.[21] As the operating net-income increases, risk-aversion and non-diversifiable risk matter less for the leader since the payoff flow from managing the business has an increasingly positive impact on the value function. Therefore, when the operating net-income becomes sufficiently large, the follower's value function is lowered more than the leader's value function as we move from complete to incomplete markets.

In line with intuition, as technology 2 is more likely to arrive, preemption will occur at a higher level of operating net-income. Specifically, from Panel B and D we observe that preemption under complete markets for $\lambda = 0.15$ occurs at a level of operating net-income where it is more profitable to be the leader than to be the follower under incomplete markets. This, *ceteris paribus*, results in

[20] However, because the follower is forward looking, she has a more indirect exposure to movements in the operating net-income through the option value function $G(y)$.

[21] This holds, of course, only up to the point where the follower's option value reaches its intrinsic value and the option's *Delta* coincides with the option's *Delta* of the leader's value function.

accelerated investment by the leader under incomplete markets.[22]

The numerical analysis in Figure 1, illustrates, that the implications of technological innovation on the optimal strategic investment timing behavior in the presence of non-diversifiable risk is ambiguous compared to complete markets. This result has ramifications for individuals contemplating to become entrepreneurs in markets where technological innovation plays an important role.

## IV.2. A Second-mover Advantage

Next, we consider a value of the arrival intensity $\lambda \in [\frac{rD_{10}}{D_{21}-D_{12}}, \frac{rD_{10}}{D_{22}-D_{12}}) = [0.2667, 0.5714)$. Panel A and C in Figure 2 show the value functions for $\lambda = 0.27$ under complete and incomplete markets, respectively. Panel B and D show a similar picture for $\lambda = 0.40$. In this situation, the arrival likelihood of technology 2 is sufficiently large such that the leader and the follower curve do not intersect, i.e., no preemption equilibrium will occur. For all values of the operating net-income, it is more optimal to invest as the follower in technology 2, than to invest as the first-mover in technology 1, i.e., there exists a second-mover advantage.

As in Huisman and Kort (2004) an attrition equilibrium arises when the leader and the waiting curve intersect each other, see also Hendricks, Weiss, and Wilson (1988). At this attrition point (illustrated by the vertical dotted line(s)), it is more profitable for one of the entrepreneurs to invest than to engage in joint waiting until technology 2 arrives, *but* it is still more profitable to invest

[22] The valuation of the leader curve slightly increases as the arrival intensity of technology 2 increase. As the arrival intensity increases, technology 2 is expected to arrive sooner and this should mainly benefit the follower. However, this phenomenon happens because, as the arrival intensity of technology 2 increases then so does the investment threshold, $\bar{y}_{11}$, when the follower optimally invests in technology 1. This implies that the leader receives the flow payoff $D_{10}Y_t$ for a longer period of time given technology 2 does not arrive and that has a slight upward impact on the valuation of the leader curve.

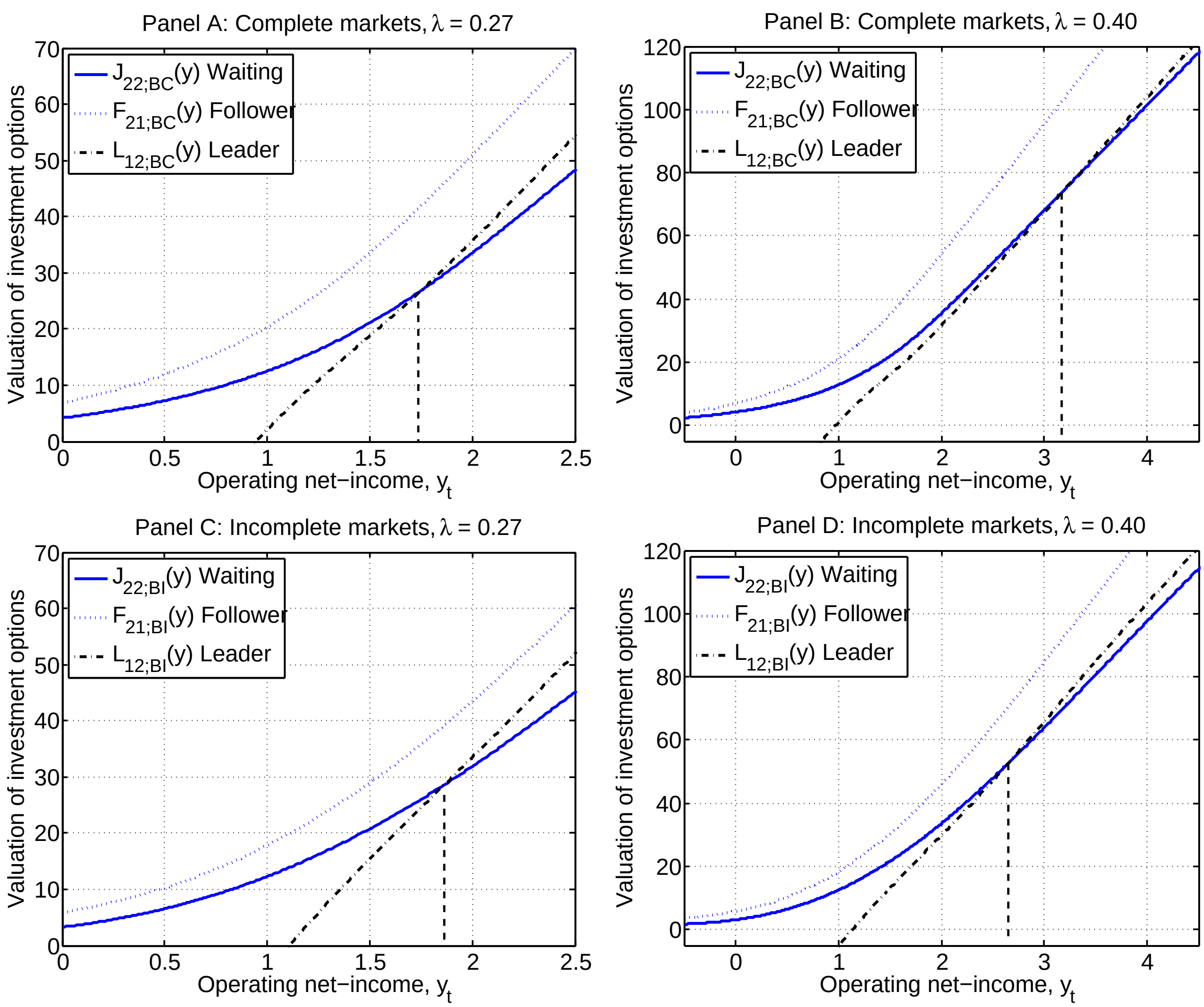


**Figure 2. Option value functions.** Panel A and B show the value functions under complete markets for $\lambda = 0.27$ and $\lambda = 0.40$ and Panel C and D show the value functions under incomplete markets for $\lambda = 0.27$ and $\lambda = 0.40$, respectively. The dash-dotted line denotes the value function for the leader, the dotted line denotes the value function for the follower, the solid line denotes the value function for joint waiting for technology 2. The other parameters are given as follows: project value drift $\alpha_y = 0.05$, project value volatility $\sigma_y = 30\%$, risk-free interest rate $r = 8\%$, investment costs $I = 50$, risky asset drift $\mu_s = 0.10$, risky asset volatility $\sigma_s = 20\%$, risk-aversion $\gamma = 0.25$, correlation between operating net-income and risky asset $\rho = 0.15$. The income factors are, $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

second and wait for the arrival of technology 2.[23]

From Figure 2 we observe, that as technology 2 is more likely to arrive over the next instant, it becomes more valuable for the entrepreneurs to engage in joint waiting for technology 2. This pushes the waiting curve up relative to the leader curve such that the attrition point occurs later under both market settings. Note also, that the differential equation defining the value function for joint waiting (Proposition 10) is identical to that defining the valuation of and the follower's value function (Proposition 6) apart from the income factor. Therefore, the impact of risk-aversion and non-diversifiable risk on the value function for joint waiting, is similar to that discussed in Section IV.1.2 for the follower's value function. Hence, the value function for joint waiting is lowered relatively more than the leader's value function, as we move from complete to incomplete markets, for increasing values of the operating net-income. As for the follower's value function, the opposite situation occurs for decreasing values of the operating net-income.

For $\lambda = 0.27$, the attrition point occurs at a level of operating net-income, where the leader's value function is lowered relatively more than the value function for joint waiting thereby resulting in delayed investment under incomplete markets. In contrast, when $\lambda = 0.40$ the attrition point occurs at a level of operating net-income where the waiting curve suffers relatively more than the leader curve thereby leading to accelerated investment under incomplete markets.

[23]Since the aim of the analysis is to understand when investment should optimally take place rather than to differentiate between which of the entrepreneurs that will invest first and given that the entrepreneurs are assumed to be similar, it is for simplicity assumed, that with probability one half, each of the entrepreneurs will invest as the leader. The attrition equilibrium represents a situation, where one of the entrepreneurs is better off investing in the existing technology (technology 1) to become the leader rather than to engage in joint waiting for the arrival of technology 2. We may think of such a situation in the market for zero-emission vehicles (cf. the anecdotal evidence in Section II.6) where new innovations are constantly being developed, but the time it takes for them to reach the market as operational technologies is highly uncertain, so eventually, one may be better off investing in the existing available technology.

### IV.3. Waiting for Technology 2

Next, we consider a value of the arrival intensity $\lambda \in [\frac{rD_{10}}{D_{22}-D_{12}}, \infty) = [0.5714, \infty)$. Panel A and B in Figure 3 illustrate the situation for $\lambda = 2.95$ under complete and incomplete markets, respectively.[24]

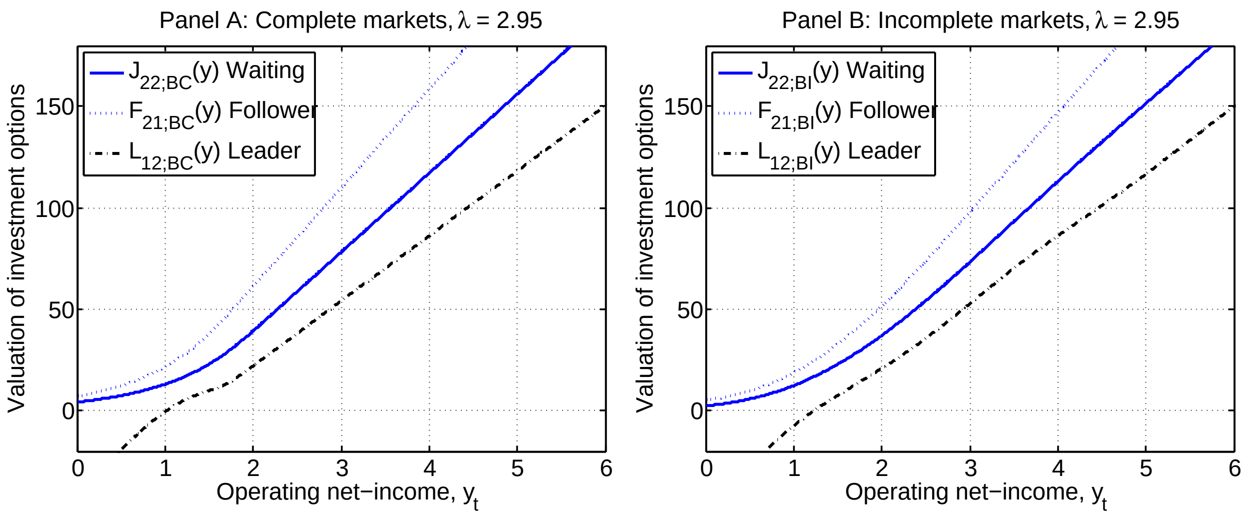


**Figure 3. Option value functions.** Panel A and B show respectively the option value functions under complete and incomplete markets. The dash-dotted line denotes the value function for the Leader, the dotted line denotes the value function for the Follower, the solid line denotes the value function for joint waiting for technology 2. The other parameters are given as follows: project value drift $\alpha_y = 0.05$, project value volatility $\sigma_y = 30\%$, risk-free interest rate $r = 8\%$, investment costs $I = 50$, drift risky asset $\mu_s = 0.10$, volatility risky asset $\sigma_s = 20\%$, risk-aversion $\gamma = 0.25$, correlation between project value and risky asset $\rho = 0.15$. The income factors are, $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

For this value of the arrival intensity, both entrepreneurs should optimally wait until technology 2 arrives. This occurs because the leader's value function is below the value function for joint waiting for all values of the operating net-income. The primary difference as we move from complete to incomplete markets is that the value functions decrease due to presence of risk-aversion and non-diversifiable risk.

[24]The value of $\lambda = 2.95$ is chosen purely for illustrative purposes since for, e.g., $\lambda = 0.58$ the waiting curve and the leader curve will be very close together making it hard to visualize any differences in the figure.

### IV.4. Empirical Implications regarding Investment Timing

The numerical analysis above has shown that significant differences can occur regarding the optimal strategic investment behavior between under-diversified individuals such as entrepreneurs who own a substantial share of their entrepreneurial business and well-diversified individuals or companies. Specifically, the degree of technological innovation as well as risk-attitude in conjunction with non-diversifiable risk appear to play an important role in determining the optimal investment timing. This leads to empirical implications regarding optimal investment timing for individuals contemplating to become entrepreneurs.

For the set of parameters used in the numerical analysis, when it is optimal for one of the entrepreneurs to invest and become the leader (cf. Figure 1–2), the predominant prediction is that technology adoption by an under-diversified risk-averse entrepreneur should occur sooner under incomplete markets compared to a well-diversified individual or company. This is consistent with other theoretical work that entrepreneurs tend to promote new innovations and technologies, e.g., Romer (1990) and Schmitz (1989) by way of imitation by established companies and Quadrini (2009) for a review. According to the model, a possible explanation for such behavior (at the micro-level) is driven by optimality concerns and risk-aversion: entrepreneurs may take strategic aspects over uncertain future technological innovations and their exposure to non-diversifiable income risk into account prior to investing.

Moreover, the numerical analysis has shown that by accounting for strategic aspects and arrival of future technological innovations, the investment timing may, in the presence of non-diversifiable risk, be accelerated compared to complete markets, in contrast to the findings in Miao and Wang (2007).

### IV.5. Analysis of the Optimal Portfolio Policy

A number of studies in the literature have documented the importance of non-diversifiable risk for individuals' portfolio choice (e.g., Merton (1971), Bodie, Merton, and F. (1992) and Viceira (2001)).

The notion is that in the presence of uninsurable risk, individuals will adjust their risky asset holdings ("hedging demand") to partially hedge against unfavorable movements in this risk factor.[25] The numerical analysis complements this literature and sheds new light onto factors which affect the hedging demand and thus optimal portfolio choice. Specifically, strategic considerations regarding entrance of other individuals and business specific factors such as technological innovation are shown to have substantial implications for the optimal portfolio choice under incomplete markets.

Panel A and B in Figure 4 illustrate the optimal portfolio choice as a function of the operating net-income for the follower and the leader for varying levels of technological innovation and incomplete hedging. In anticipation of a more efficient technology arriving sooner and for higher correlation between the operating net-income and the risky asset, the more the follower should reduce the risky asset holdings as the operating net-income increases. Specifically, as the option's *Delta* becomes more sensitive to changes in the operating net-income, the follower optimally hedges the larger variations that can occur in the certainty-equivalent valuation of the non-traded investment opportunity by lowering the exposure to the risky asset already prior to exercising the investment option. This result has implications for prospective entrepreneurs contemplating to start up their own business in markets where technological innovation plays a fundamental role.

In the same vein, the leader should also adjust the portfolio allocation in the risky asset as the arrival intensity of technology 2, the amount of hedging and the operating net-income increases in anticipation of sooner entry by the follower. Once the follower invests, the leader's operating net-income from managing the business is adversely affected and the exposure to non-diversifiable risk is reduced. Therefore, the leader should increase the holdings in the risky asset already prior to investment by the follower, however still at levels below the myopic mean-variance portfolio consistent with findings in Hongyan and Nofsinger (2009) and Heaton and Lucas (2000). This gives

[25] In the numerical analysis, the correlation between innovations to the risky asset and the operating net-income is assumed to be positive ($\rho = 0.15$) and therefore, the entrepreneurs will attempt to hedge the uninsurable income risk by reducing the proportion of wealth invested in risky assets relative to the mean-variance portfolio. This is consistent with empirical evidence in, e.g., Heaton and Lucas (2000) who find a positive correlation of 0.14 between the quarterly growth rate of real non-farm proprietary income for U.S. entrepreneurial businesses and the CRSP value-weighted stock return.

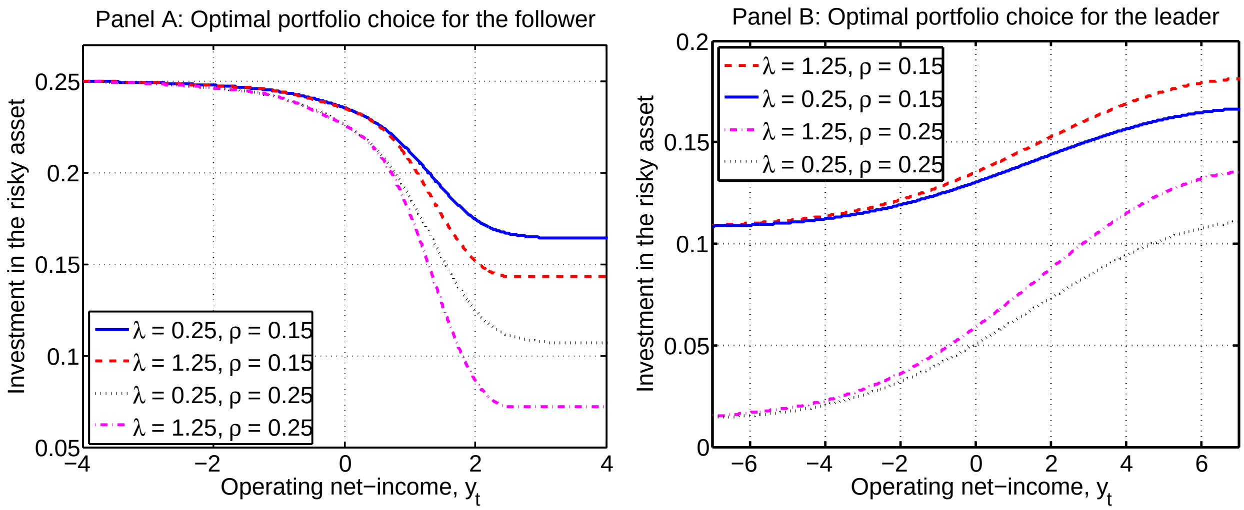


**Figure 4. Optimal portfolios.** Panel A and B exhibit the optimal portfolio for the follower and leader divided by the initial wealth as a function of the operating net-income respectively, for selected parameters. The solid line is for $\rho = 0.15, \lambda = 0.25$, the dashed line $\rho = 0.15, \lambda = 1.25$, the dotted line $\rho = 0.25, \lambda = 0.25$ and the dashed-dotted line $\rho = 0.25, \lambda = 1.25$. The other parameters are set as follows: project value drift $\alpha_y = 0.05$, project value volatility $\sigma_y = 30\%$, risk-free interest rate $r = 8\%$, investment costs $I = 50$, drift risky asset $\mu_s = 0.10$, volatility risky asset $\sigma_s = 20\%$, risk-aversion $\gamma = 0.25$, initial wealth $W_0 = 100$. The income factors are given as follows: $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

rise to another empirical prediction. For a given amount of non-diversifiable risk, entrepreneurs managing a business in profitable environments where technological innovation and entrance by other individuals are more likely to occur should have more wealth invested in the financial market compared to their peers in environments where technological innovation and entrance by other individuals are less likely to occur.

## IV.6. Comparative Statics Analysis

This section discusses comparative statics regarding the optimal investment timing decision and optimal portfolio choice. To facilitate the discussion, the set of parameters specified in Figure 1-3 are kept fixed, except for the parameter of interest discussed below.

First, if the entrepreneurs are more risk-averse ($\gamma = 0.45$), preemption will occur later relative to the setting in Figure 1 with $\lambda = 0.15$. But preemption will still occur sooner compared to complete markets. This is intuitive, higher risk-aversion implies that the operating net-income has to be higher to make it profitable to invest as the leader. Furthermore, investment by the follower will occur sooner compared to the situation with $\lambda = 0.15$. Higher risk-aversion implies that the certainty-equivalent gain under incomplete markets is reduced compared to the setting with $\gamma = 0.25$, i.e., the gain from waiting for arrival of technology 2 is further reduced, and that leads to accelerated investment compared to the setting with lower risk-aversion.[26]

Higher project volatility ($\sigma_y = 0.40$) implies that the entrepreneurs are also exposed to greater non-diversifiable risk and that has an overall negative impact on the value functions. Specifically, the leader's value function is lowered more for decreasing operating net-income thus resulting in later preemption compared to the incomplete market setting with $\sigma_y = 0.30$ (for both cases $\lambda = 0.02, 0.15$). Similar to increasing risk-aversion, greater project volatility ($\sigma_y = 0.40$) reduces the certainty-equivalent gain from arrival of technology 2 and that has a larger negative impact on the option value compared to the project value when $\lambda = 0.15$. This results in sooner investment by the follower in technology 1 relative to the incomplete market setting in Figure 1. In contrast, for $\lambda = 0.02$ the certainty-equivalent gain matters less for the option valuation and the project value is therefore reduced more than the option value resulting in delayed investment.

Similar to the optimal investment timing, the optimal portfolio choice of the entrepreneurs is also sensitive to the profitability of adopting a given technology. This is captured by the income factors (i.e., the $D_{ij}$'s). A higher income factor implies higher expected drift in the operating net-income, but it also implies higher exposure to non-diversifiable risk. For instance, a higher income factor $D_{21}$, would from the followers perspective lead to higher valuation of being the follower and more importantly a higher option's *Delta* thereby resulting in a larger hedging demand and thus a smaller amount of wealth invested in the risky asset. Hence, it is important to take strategic considerations

[26] For the situation where $\lambda = 0.02$ the predictions in Figure 1 continue to hold relative to complete markets. However, compared to the incomplete market setting, preemption when $\gamma = 0.45$ occurs later due to higher risk-aversion and the follower also invests later since the project value is reduced even further relative to the option value in this case.

regarding technology adoption by other individuals and arrival of future technological innovations into account when investing in the financial markets. But the relative profitability of managing the business with a given technology compared to its competitors, also matters profoundly when deciding how much wealth to allocate to the financial markets in order to hedge the non-diversifiable risk.

## V. Concluding Remarks

I have studied the implications of technological innovation and strategic considerations regarding technology adoption on optimal investment timing and valuation of the associated investment opportunities. According to the model, before individuals (contemplating to become entrepreneurs) decide whether to invest in a business project using a current available technology or whether to wait until a more efficient technology may be available for adoption, they should take into account: (a) the likelihood of such a better technology arriving; (b) potential entrance by other individuals and the impact on profitability; (c) any non-diversifiable (income) risk surrounding the business project. Failure to account for these elements when deciding to invest, may lead individuals to overestimate the value attached to the investment opportunity and subsequently lead to suboptimal investment behavior. A numerical analysis has shown that these considerations have important implications not only for optimal investment timing but also for the optimal portfolio choice for both prospective and current entrepreneurs. In future research, it would be interesting to investigate to what extent these predictions hold empirically.

To study the implications of non-diversifiable risk, technological innovation and strategic aspects regarding technology adoption on investment timing under uncertainty in the simplest possible way, the model has a number of limitations which could be interesting to address in future research. One extension would be to consider a power utility function which is arguably more realistic than the exponential utility function. The advantage of working with an exponential utility function is that it helps reduce the dimension of the problem since it does not capture wealth effects. A power utility function, on the other hand, would introduce wealth effects into the model and the non-linear ODE's which determine the (option) value functions would become non-linear partial differential equations subject to a free-boundary value and this is significantly more challenging to solve numerically. Another extension relates to the innovation process. In future research it would be interesting to relax the exogenous innovation assumption and make the innovation process endogenous within the model.

A final interesting extension would be to pursue a calibration of the model parameters to actual data. However, challenges in defining and measuring the precise role of an entrepreneur (see, e.g., Wong, Ho, and Autio (2005)) leads to difficulties in calibrating such a model to actual data. Moreover, there is generally little available data for measuring the amount of non-diversifiable income risk, the degree of technological innovations, the risk attitude of entrepreneurs, etc. An attempt to calibrate an entrepreneurial model is done in Heaton and Lucas (2004). They discuss issues related to the calibration of their parameters and also have to resort to qualitative guidance on parameter choices for risk-aversion, idiosyncratic risk and so on. In this paper, I have not pursued a "calibration" of the model parameters since the primary focus was on the qualitative implications.

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# Appendix A

### Proof of Proposition 1

We begin by considering the situation after investment has taken place. Let, $\pi_t$, denote the amount invested in the risky asset measured in units of the consumption good (i.e., $\pi_t$, represents the amount of cash in the risky asset) and the remainder is invested in the risk-free bond. After investment in technology $N_i$ given the competitor has invested in technology $N_j$, the entrepreneur receives a flow payoff equal to $D_{N_iN_j}Y_t$ and consumes $c_t$. This means, that the wealth dynamics after investment to the entrepreneurs equals,

$$\begin{aligned} dW_t &= \pi_t \frac{dS_t}{S_t} + r(W_t - \pi_t)dt + D_{N_iN_j}Y_t dt - c_t dt \\ &= \Big(rW_t + \pi_t(\mu_S - r) + D_{N_iN_j}Y_t - c_t\Big)\, dt + \pi_t \sigma_S dB_t. \end{aligned} \tag{A.1}$$

Denote by $J(w, y)$ the value function to the entrepreneur after option exercise. After option exercise, the wealth dynamics, $w$, but also the project value, $y$, affect the value function after option exercise because the entrepreneurs are exposed to non-diversifiable risk from managing the business which cannot be hedged by trading in the risky financial asset.[27] An application of Ito's formula yields

[27] As shown in Miao and Wang (2007), this is different to the lump-sum case where the exercise of the investment option generates an exit from incomplete markets.

the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\begin{aligned}\beta J(w,y) &= \max_{\pi,C} U(c) + J_w(w,y)\Big(rw + \pi(\mu_S - r) + D_{N_iN_j}y - c\Big) + J_y(w,y)\alpha_y \\ &+ \frac{\pi^2\sigma_S^2}{2}J_{ww}(w,y) + \frac{\sigma_y^2}{2}J_{yy}(w,y) + \rho\sigma_y\pi\sigma_S J_{wy}(w,y) \qquad \text{(A.2)}\end{aligned}$$

which is subject to the transversality condition $\lim_{T\to\infty} E[e^{-\beta T}J(W_T,Y_T)] = 0$. For simplicity, I have omitted the time-index. Now conjecture that the value function takes the following form,

$$J(w,y) = -\frac{1}{\gamma r}\exp\left(-\gamma r\left(w + f(y;D_{N_iN_j}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right)\right) \qquad \text{(A.3)}$$

for some function, $f(y;D_{N_iN_j})$, which have to be determined and where $\eta = \frac{\mu_S - r}{\sigma_S}$, denotes the Sharpe-ratio on the risky asset. Deriving the first-order conditions with respect to the portfolio and consumption gives,

$$\begin{aligned}0 &= J_w(w,y)(\mu_S - r) + \sigma_S^2\pi J_{ww}(w,y) + J_{wy}(w,y)\rho\sigma_y\sigma_S \\ \pi^* &= \frac{-J_w(w,y)(\mu_S - r)}{J_{ww}(w,y)\sigma_S^2} + \frac{-J_{wy}(w,y)\rho\sigma_y}{J_{ww}(w,y)\sigma_S} \qquad \text{(A.4)}\end{aligned}$$

and

$$U'(c) = J_w(w,y) \Leftrightarrow c^* = -\frac{1}{\gamma}\log(J_w(w,y)) \qquad \text{(A.5)}$$

By relying on the conjecture, the derivatives of the value function can be written as follows,

$$\begin{aligned}J_w(w,y) &= -\gamma r J(w,y) \qquad J_{ww}(w,y) = (\gamma r)^2 J(w,y) \\ J_{wy}(w,y) &= (\gamma r)^2 f'(y;D_{N_iN_j})J(w,y) \\ J_y(w,y) &= -\gamma r f'(y;D_{N_iN_j})J(w,y) \\ J_{yy}(w,y) &= -\gamma r f''(y;D_{N_iN_j})J(w,y) + (\gamma r)^2 f'(y;D_{N_iN_j})^2 J(w,y) \qquad \text{(A.6)}\end{aligned}$$

and we obtain the optimal portfolio and consumption policy as,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} f'(y; D_{N_iN_j}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{N_iN_j}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.7}$$

Plugging them back into the HJB equation and simplifying we obtain the following non-linear ordinary differential equation (ODE) for the function, $f(y; D_{N_iN_j})$,

$$\begin{aligned}&\frac{\sigma_y^2}{2} f''(y; D_{N_iN_j}) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) f'(y; D_{N_iN_j})^2 + (\alpha_y - \rho\sigma_y\eta) f'(y; D_{N_iN_j}) \\ &-r f(y; D_{N_iN_j}) + D_{N_iN_j} y = 0\end{aligned} \tag{A.8}$$

which has to be solved subject to the transversality condition for the value function stated above. Ruling out speculative bubbles in the project value (cf. Dixit and Pindyck (1994)) we obtain the solution given in the proposition.

## Proof of Proposition 2

The situation after option exercise is derived in Proposition 1 by setting $D_{N_iN_j} = D_{22}$. Thus the project value is given by,

$$f(y; D_{22}) = \frac{D_{22}}{r} y + \frac{(\alpha_y - \rho\sigma_y\eta) D_{22}}{r^2} - \frac{\gamma\sigma_y^2(1-\rho^2) D_{22}^2}{2r^2} \tag{A.9}$$

and the optimal portfolio and consumption policy are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} f'(y; D_{22}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{22}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.10}$$

To analyze the situation before option exercise, we note that the wealth dynamics before investment to the entrepreneurs is given by,

$$dW_t = \big(rW_t + \pi(\mu_S - r) - c_t\big)\, dt + \pi\sigma_S dB_t \tag{A.11}$$

Before investing neither of the entrepreneurs receive a flow payoff since by assumption $D_{0N_j} = 0$. Similar to derivations for the project value derived in Proposition 1, we conjecture that the value function takes the following form,

$$V(w,y) \quad = \quad -\frac{1}{\gamma r}\exp\left(-\gamma r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right)\right) \tag{A.12}$$

for some function $g(y)$ which have to be determined and where $\eta = \frac{\mu_s - r}{\sigma_S}$ again denotes the Sharpe-ratio on the risky asset. The derivations are similar to those in the proof of Proposition 1. Hence, we obtain the optimal portfolio and consumption,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S}g'(y) \quad \text{and} \quad c^* = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.13}$$

and where the function $g(y)$ satisfies the following non-linear ODE,

$$\frac{\sigma_y^2}{2}g''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)g'(y)^2 + (\alpha_y - \rho\sigma_y\eta)g'(y) - rg(y) = 0 \tag{A.14}$$

The function $g(y)$ represents the option value function to the entrepreneurs. The function has to be solved numerically subject to the value matching and smooth-pasting condition,

$$g(\bar{y}_{22}) \quad = \quad f(\bar{y}_{22}; D_{22}) - I \tag{A.15}$$

$$g'(\bar{y}_{22}) \quad = \quad f'(\bar{y}_{22}; D_{22}) = \frac{D_{22}}{r} \tag{A.16}$$

and the lower boundary condition, $\lim_{y\to-\infty} g(y) = 0$. The optimal investment threshold $\bar{y}_{22}$ also has to be found numerically. The reason for the three boundary conditions is that we have to jointly determine the endogenous investment threshold $\bar{y}_{22}$ together with the option value function $g(y)$.

The option value function can take the following form,[28]

$$J_{22;AI}(y) = \begin{cases} g(y) & \text{if } y \in (-\infty, \bar{y}_{22}) \\ f(y; D_{22}) - I & \text{if } y \in (\bar{y}_{22}, +\infty). \end{cases}$$

The optimal portfolio and consumption policy after investment are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{22}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{22}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.17}$$

## Proof of Proposition 3

The situation after option exercise is derived in Proposition 1 by setting $D_{N_i N_j} = D_{21}$. Thus the project value is given by,

$$f(y; D_{21}) \quad = \quad \frac{D_{21}}{r} y + \frac{(\alpha_y - \rho \sigma_y \eta) D_{21}}{r^2} - \frac{\gamma \sigma_y^2 (1 - \rho^2) D_{21}^2}{2r^2} \tag{A.18}$$

and the optimal portfolio and consumption policy after investment are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{21}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{21}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.19}$$

whereas before investment they take the form,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} g'(y) \quad \text{and} \quad c^* = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.20}$$

The derivations to obtain the follower's option value function when technology 2 is available is similar to the proof of Proposition 2 by replacing the income factor $D_{22}$ with $D_{21}$ and where the optimal investment threshold for the follower in technology 2 is instead denoted by $\bar{y}_{12}$. The boundary

[28] The value function is denoted by $J_{22;AI}(y)$ where $J$ refers to joint investment, the subscript "22" refers to the income factor $D_{22}$ to the operating net-income process the entrepreneurs receive and $A$ refers to the situation after technology 2 has arrived and $I$ that it is the value function under incomplete markets.

conditions are given analogously.

## Proof of Proposition 4

After option exercise the leader receives the flow payoff $D_{10}Y_t$ up to the moment when the follower invests in technology 2. From that point and onwards the leader's flow payoff is reduced to $D_{12}Y_t$. Before the follower optimally invests in technology 2 at the threshold $\bar{y}_{12}$, the leader has wealth dynamics given by,

$$dW_t \quad = \quad \big(rW_t + \pi(\mu_S - r) + D_{10}Y_t - c_t\big)\, dt + \pi\sigma_S dB_t \tag{A.21}$$

Deriving the HJB equation following similar steps as in the earlier proofs, the value function for the leader satisfies the following non-linear ODE,

$$\frac{\sigma_y^2}{2} g''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) g'(y)^2 + (\alpha_y - \rho\sigma_y\eta) g'(y) - r g(y) + D_{10} y = 0 \tag{A.22}$$

subject to the lower boundary condition, $\lim_{y \to -\infty} g(y) = f(y; D_{10})$ and the value-matching condition $g(\bar{y}_{12}) = f(\bar{y}_{12}; D_{12})$ where the function $f(\cdot)$ is given in Proposition 1. The reduction in the flow payoff to the leader, when the follower invests in technology 2, is captured in the value-matching condition. The optimal portfolio and consumption policy of the leader before the follower has invested are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} g'(y) \quad \text{and} \quad c^* = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.23}$$

and after the follower has invested they are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} f'(y; D_{12}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{12}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.24}$$

### Proof of Proposition 5

The proof is similar to that for Proposition 1. It follows by inserting the income factor $D_{11}$ and the optimal portfolio policy of the entrepreneurs is similar to that in Proposition 2.

### Proof of Proposition 6

Denote the value function to the follower before option exercise by, $V(w, y)$, where,

$$V(w,y) \quad = \quad -\frac{1}{\gamma r}\exp\left(-\gamma r\left(w+G(y)+\frac{\eta^2}{2\gamma r^2}+\frac{\beta-r}{\gamma r^2}\right)\right) \tag{A.25}$$

and where the function, $G(y)$, represents the follower's option value before technology 2 has arrived. The wealth dynamics of the follower before investment is given by,

$$dW_t \quad = \quad \left(rW_t+\pi(\mu_S-r)-c_t\right)dt+\pi\sigma_S dB_t+\lambda\left(F_{21;AI}(y)-G(y)\right)dt \tag{A.26}$$

Note that, at any given point in time, the state of the world may change, in that technology 2 will arrive with probability $\lambda dt$ thus making technology 1 inferior from that point and onwards. This means, that the entrepreneur, at any instant, can experience an expected capital gain corresponding to $F_{21;AI}(y) - G(y)$ with probability $\lambda dt$ from the arrival of technology 2. Note, also that the terms $F_{21;AI}(y)$ and $G(y)$ are actual values, i.e., they are not utility terms. This expected positive shift in wealth materializing at some future time $T$ has to be reflected in the wealth dynamics. That explains the new term, $\lambda\left(F_{21;AI}(y) - G(y)\right)dt$ in the wealth dynamics.

Accounting for the arrival of technology 2 with intensity $\lambda$ we obtain the HJB equation,

$$\begin{aligned}\beta V(w,y) \quad &= \quad \max_{\pi,C} U(c)+V_w(w,y)\left(rw+\pi(\mu_S-r)+\lambda\left(F_{21;AI}(y)-G(y)\right)-c\right)\\ &+ \quad V_y(w,y)\alpha_y+\frac{\pi^2\sigma_S^2}{2}V_{ww}(w,y)+\frac{\sigma_y^2}{2}V_{yy}(w,y)+\rho\sigma_y\pi\sigma_S V_{wy}(w,y)\end{aligned} \tag{A.27}$$

subject to the transversality condition $\lim_{T\to\infty} E[e^{-\beta T}V(W_T, Y_T)] = 0$. Following similar steps as earlier we derive the following non-linear ODE,

$$\frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) - (r+\lambda)G(y) + \lambda F_{21;AI}(y) = 0 \tag{A.28}$$

where the function $F_{21;AI}(y)$ was determined in Proposition 3.

By inserting the expression for $F_{21;AI}(y)$ we can derive the boundary conditions. The lower boundary condition $\lim_{y\to-\infty} G(y) = 0$ represents that the follower's option value loses its value as the operating net-income approaches minus infinity. The upper growth condition follows as the particular solution to the system of ODE's we obtain by inserting the expression for $F_{21;AI}(y)$. Therefore, the upper growth condition becomes $\lim_{y\to+\infty} G(y) = Ay + B$ where,

$$\begin{aligned} A &= \frac{\lambda D_{21}}{r(r+\lambda)} &\text{(A.29)}\\ B &= -\frac{\gamma r\sigma_y^2(1-\rho^2)\lambda^2 D_{21}^2}{2r^2(r+\lambda)^3} + \frac{(\alpha_y - \rho\sigma_y\eta)\lambda D_{21}}{(r+\lambda)^2 r} + \frac{\lambda(\alpha_y - \rho\sigma_y\eta)D_{21}}{(\lambda+r)r^2} \\ &- \frac{\lambda\gamma\sigma_y^2(1-\rho^2)D_{21}^2}{(r+\lambda)2r^2} - \frac{\lambda}{(r+\lambda)}I &\text{(A.30)}\end{aligned}$$

To get insight about the upper boundary condition, it is useful to consider it under complete markets. Let $\lim_{y\to\infty} G^c(y)$ denote the expected value under complete markets of receiving the flow payoff $D_{21}Y_t$ in perpetuity starting from time $T$ (when technology 2 arrives) and paying the investment costs $I$. Said quantity can be computed as follows,

$$\begin{aligned}\lim_{y\to\infty} G^c(y) &= E_{Y_0=y}\left[e^{-rT}\left(\frac{D_{21}}{r}Y_T + \frac{(\alpha_y - \sigma_y\eta)D_{21}}{r^2} - I\right)\right] \\ &= \frac{\lambda D_{21}}{r(r+\lambda)}y + \frac{(\alpha_y - \sigma_y\eta)\lambda D_{21}}{(r+\lambda)^2 r} + \frac{\lambda(\alpha_y - \sigma_y\eta)D_{21}}{(\lambda+r)r^2} - \frac{\lambda}{(r+\lambda)}I\end{aligned}$$

since $E_{Y_0=y}(Y_t) = y + (\alpha_y - \sigma_y\eta)t$ under complete markets and because $T$ is exponentially distributed with intensity $\lambda$. The expression for $\lim_{y\to\infty} G^c(y)$ resembles the upper growth condition in Proposition 6 except for the two remaining terms, that incorporate the notion of risk-aversion and incomplete hedging. Thus, the upper growth condition in Proposition 6 represents the expected net-present value of the flow payoff $D_{21}Y_t$ accruing to the follower from investing at time $T$ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging. The optimal portfolio and consumption policy of the follower before investment are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.31}$$

for $y \in (-\infty, \infty)$.

### Proof of Proposition 7

The problem is similar to that in Proposition 6 except that the upper boundary condition changes. The follower invests at the threshold $\bar{y}_{11}$ which has to be determined as part of the problem. Therefore the value-matching and smooth-pasting conditions become, $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11}) - I$ and $G'(\bar{y}_{11}) = f'(\bar{y}_{11}; D_{11})$. The optimal portfolio and consumption policy of the follower before investment are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.32}$$

for $y \in (-\infty, \bar{y}_{11})$ and after investment in technology 1 they are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{11}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{11}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.33}$$

for $y \in (\bar{y}_{11}, \infty)$.

### Proof of Proposition 8

Denote the value function for the leader (before the follower optimally exercises the option to invest in technology 2 at the threshold $\bar{y}_{12}$) by $V(w,y)$ where,

$$V(w,y) \quad = \quad -\frac{1}{\gamma r}\exp\left(-\gamma r\left(w+G(y)+\frac{\eta^2}{2\gamma r^2}+\frac{\beta-r}{\gamma r^2}\right)\right) \tag{A.34}$$

and where $G(y)$ represents the value function. The leader's wealth dynamics takes the following form,

$$dW_t = \big(rW_t + \pi(\mu_S - r) + D_{10}Y_t - c_t\big)\,dt + \pi\sigma_S dB_t + \lambda\left(L_{12;AI}(y) - G(y)\right)dt \tag{A.35}$$

Note, that we assume immediate investment in technology 1 by the leader which means that she receives a flow payoff equal to $D_{10}Y_t$ accruing to the wealth at any given time. Moreover, the leader may experience (at any given point in time with probability $\lambda dt$) an expected shift in wealth corresponding to $L_{12;AI}(y) - G(y)$ similar to that of the follower which happens when technology 2 arrives. Deriving the usual HJB equation we obtain that the non-linear ODE defining the leader's value function is given by,

$$\begin{aligned} rG(y) \quad &= \quad D_{10}y + \lambda\left(L_{12;AI}(y) - G(y)\right) + (\alpha_y - \rho\sigma_y\eta)G'(y) \\ &- \quad \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + \frac{\sigma_y^2}{2}G''(y) \end{aligned} \tag{A.36}$$

and the expressions for $L_{12;AI}(y)$ is determined in Proposition 4.

As the operating net-income approaches minus infinity, it is not optimal for the follower to invest regardless of the technology available and therefore the value function for the leader converges to the project value $f(y;D_{10})$. By inserting the expressions for $L_{12;AI}(y)$ into the non-linear ODE above we can derive the upper growth condition as the particular solution. The upper growth condition equals $\lim_{y\to\infty} G(y) = Ay + B$ where

$$
\begin{aligned}
A &= \frac{D_{10}}{(r+\lambda)} + \frac{\lambda D_{12}}{r(r+\lambda)} \\
B &= -\frac{\gamma r \sigma_y^2 (1-\rho^2)\left(\frac{D_{10}}{(r+\lambda)} + \frac{\lambda D_{12}}{r(r+\lambda)}\right)^2}{2(r+\lambda)} + \frac{(\alpha_y - \rho\sigma_y\eta)\left(\frac{D_{10}}{(r+\lambda)} + \frac{\lambda D_{12}}{r(r+\lambda)}\right)}{(r+\lambda)} \\
&+ \frac{\lambda(\alpha_y - \rho\sigma_y\eta)D_{12}}{(\lambda+r)r^2} - \frac{\lambda\gamma\sigma_y^2(1-\rho^2)D_{12}^2}{(r+\lambda)2r^2}
\end{aligned}
$$

To obtain intuition about the upper boundary condition it is again useful to consider it under complete markets. Under complete markets the growth condition for the leader contains the expected present value of receiving the flow payoff $D_{12}Y_t$ in perpetuity from time $T$ when the follower invests in technology 2. The leader invest immediately and thus receives a flow payoff $D_{10}Y_t$ but only up to the moment when the follower invests in technology 2. Hence, the growth condition also contains the expected present value of receiving the flow payoff $D_{10}Y_t$ up to time $T$. Denoting said quantity by $\lim_{y\to\infty} G^c_{D_{10}}(y)$ it follows as,

$$
\begin{aligned}
\lim_{y\to\infty} G^c_{D_{10}}(y) &= E_{Y_0=y}\left[\int_0^T e^{-rt} D_{10} Y_t dt\right] = E\left(E_{Y_0=y}\left(\int_0^k e^{-rt} D_{10} Y_t dt | T = k\right)\right) \\
&= \frac{D_{10}}{(r+\lambda)} y + \frac{(\alpha_y - \sigma_y\eta)D_{10}}{(r+\lambda)^2}
\end{aligned}
$$

since $E_{Y_0=y}(Y_t) = y + (\alpha_y - \sigma_y\eta)t$ under complete markets and because $T$ is exponentially distributed with intensity $\lambda$. To sum up, the upper growth condition in Proposition 8 captures the expected present value of receiving the flow payoff $D_{10}Y_t$ (from immediate investment in technology 1) up to time $T$ and from that point onwards the flow payoff $D_{12}Y_t$ in perpetuity adjusted for risk-aversion and incomplete hedging.

The optimal portfolio and consumption policy of the leader before investment of the follower are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S}G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.37}$$

for $y \in (-\infty, \infty)$.

## Proof of Proposition 9

The problem is similar to that in Proposition 8 except that the upper boundary condition changes. The follower invests at the threshold $\bar{y}_{11}$ and that results in the following value-matching condition, $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11})$. The optimal portfolio and consumption policy of the leader before the follower has invested in technology 1 are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S}G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.38}$$

for $y \in (-\infty, \bar{y}_{11})$ and after the follower has invested in technology 1 they are given by,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S}f'(y; D_{11}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{11}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.39}$$

for $y \in (\bar{y}_{11}, \infty)$.

## Proof of Proposition 10

The proof is similar to that for Proposition 6 by replacing the income factor $D_{21}$ with $D_{22}$ and the value function when technology 2 is available to be inserted in the non-linear ODE is $J_{22,AI}(y)$ instead of $F_{21,AI}(y)$. The option value function $G(y)$ therefore satisfies the following non-linear ODE,

$$rG(y) = \frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) + \lambda(J_{22;AI}(y) - G(y)) \tag{A.40}$$

on $y \in (-\infty, \infty)$ where the function $J_{22;AI}(y)$ was determined in Proposition 2.

By inserting the expression for $J_{22;AI}(y)$ we can derive the boundary conditions as the particular solutions to the system of non-linear ODE's. The lower boundary condition is similar to that in Proposition 6. The upper growth condition equals $\lim_{y\to+\infty} G(y) = Ay + B$ where

$$\begin{aligned} A &= \frac{\lambda D_{22}}{r(r+\lambda)} \\ B &= -\frac{\gamma r \sigma_y^2 (1-\rho^2)\lambda^2 D_{22}^2}{2r^2(r+\lambda)^3} + \frac{(\alpha_y - \rho\sigma_y\eta)\lambda D_{22}}{(r+\lambda)^2 r} + \frac{\lambda(\alpha_y - \rho\sigma_y\eta) D_{22}}{(\lambda+r)r^2} \\ &- \frac{\lambda\gamma\sigma_y^2(1-\rho^2)D_{22}^2}{(r+\lambda)2r^2} - \frac{\lambda}{(r+\lambda)} I \end{aligned}$$

Similar to the motivation for the boundary condition in Proposition 6, the upper growth condition in Proposition 10 represents the expected net-present value of the flow payoff $D_{22}Y_t$ accruing to both entrepreneurs from investing at time $T$ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging. The optimal portfolio and consumption policy of the entrepreneurs before investment in technology 2 are,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.41}$$

for $y \in (-\infty, +\infty)$.

## Proof of Proposition 11

The proof is similar to that for Proposition 5.

# Chapter 3

## *Managerial Incentives to Take Asset Risk*

*joint work by Marc Chesney, Jacob Strømberg and Alexander F. Wagner*

# Managerial Incentives to Take Asset Risk*

Marc Chesney† Jacob Strømberg‡ Alexander F. Wagner§

Abstract

We argue that incentives to take *equity* risk ("equity incentives") only partially capture incentives to take *asset* risk ("asset incentives"). This is because leverage, while central to the theory of risk-shifting, is not explicitly considered by equity incentives. Employing measures of asset incentives that account for leverage, we find that asset risk-taking incentives can be large compared to incentives to increase firm value. Moreover, stock holdings can induce substantial risk-taking incentives, qualifying common beliefs regarding the central role of stock options. Finally, only asset incentives explain asset risk-taking of U.S. financial institutions before the 2007/08 crisis.

JEL-Classification: G01, G28, G34

Keywords: Executive compensation; managerial incentives; risk-shifting; asset risk-taking; compound options; financial crisis; write-downs

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## I. Introduction

This paper investigates managerial incentives to take *asset* risk. A significant theoretical literature deals with these incentives, offering the central insight that managerial compensation structures should "*...take into account not only the agency relationship between shareholders and management, but also the conflicts of interests which arise in the other contracting relationships ...*" (John and John, 1993, p. 950). Besides the manager-shareholder conflict, perhaps the most important conflict arising in the corporate "nexus of contracts" (Jensen and Meckling, 1976) is the shareholder-bondholder conflict regarding "asset substitution." While shareholders want to align managerial interests with their own and also to shift risk to bondholders to some extent, they consider that asset risk-taking incentives optimally avoid excessive (firm value-reducing) agency costs of debt.

Although these ideas have long been theoretically established,[1] little is known empirically about managerial incentives to take asset risk embedded in observed compensation contracts. This is surprising, given the great relevance of these incentives in corporate finance research (in particular, for work aiming to explain asset risk-taking by firms) and discussions in practice. This paper aims to make progress on this issue by addressing two questions: *First*, how powerful are a typical manager's incentives to take asset risk and to increase firm value? *Second*, can asset risk-taking incentives add to our understanding of the cross-sectional variation of asset risk-taking?

Section I deals with the first question. We consider a CEO who receives stock and/or stock options, the two most direct means of aligning shareholder and managerial interests. To understand the embedded incentives to take asset risk, we not only model equity itself as an option on the firm's assets (Merton, 1974), but, to be consistent, we also treat stock options as compound options on the underlying asset value (Geske, 1979). This combined approach is novel to the incentives literature.

In this framework, we calculate the following incentive measures: *Asset Volatility Vega* is the dollar change of the value of a stock or stock option with respect to a 0.01 change in the asset

[1] Other theoretical papers studying how optimal incentives navigate the two conflicts include, for example, Haugen and Senbet (1981) and John, Saunders, and Senbet (2000).

return volatility, and *Asset Delta* is the dollar change of the value of a stock or stock option with respect to a one percent change in the firm value. One auxiliary contribution of this paper is that we provide a correct formula for the sensitivity of a compound option to the volatility of the underlying (departing from the formula presented in Geske (1979)).

These two quantities capture *asset incentives.* We choose this terminology to emphasize that although the manager receives pay whose value depends on equity values, ultimately we are interested in his implied incentives to influence asset values and asset risk.

By contrast, *Equity Volatility Vega* measures the incentives of the CEO to increase stock return volatility. *Equity Delta* measures the incentives to increase the stock price. These two quantities, usually calculated as in Core and Guay (2002), have been widely used in the empirical literature on risk-taking and incentives. They form the *equity incentives.*

The critical difference between asset incentives and equity incentives is that the level of debt enters explicitly only in the calculations of asset incentives. To intuitively see why this is important when aiming to compare, across CEOs, incentives to take asset risk, note that the additional asset risk-taking that causes a 0.01 change in equity return volatility depends on leverage. This reflects the idea from the theory of asset substitution that managerial asset risk-taking incentives change as leverage changes. However, leverage does not feature explicitly in the calculation of equity incentives; therefore, equity incentives can indicate asset incentives, but the cross section of equity incentives only partially captures the cross section of asset incentives. More formally, for a given CEO, asset volatility and equity volatility are linked through the elasticity of the equity value with respect to the asset value, and this elasticity varies (non-linearly) across firms as leverage varies. In the cross section of CEOs, asset incentives are, therefore, not simply a uniformly scaled version of equity incentives.

To develop this intuition and to explore the quantitative importance of the difference between the incentive measures in practice, we then consider a CEO receiving a compensation package of a given value, consisting of different combinations of stock and stock options. For an interpretation of the overall incentives to take asset risk embedded in a compensation package, we note that because the manager is risk-averse, higher exposure to firm value movements (due to a higher Asset Delta) makes the manager less willing to take asset risk. A meaningful measure of overall managerial asset substitution incentives is, therefore, given by the *Asset Incentive Ratio*, the ratio of total Asset Volatility Vega and total Asset Delta. Similarly, the *Equity Incentive Ratio* is the ratio of total Equity Volatility Vega and total Equity Delta.

This basic analysis yields four results. *First*, the Asset Incentive Ratio suggests significant asset risk-taking incentives even when a CEO is compensated only with stock. *Second*, for all combinations of stock and stock options, the Asset Incentive Ratio is significantly higher than the Equity Incentive Ratio. *Third*, as expected, the difference between the Asset Incentive Ratio and the Equity Incentive Ratio is greater when leverage is higher; however, even at 40% leverage (the average leverage of a BBB-rated firm) the differences can be substantial. *Fourth*, the higher leverage is, the less stock options add in terms of asset risk-taking incentives (compared to equity risk-taking-incentives); intuitively, at high leverage, stock itself has such strong optionality that even a CEO holding only stock already has strong incentives to take asset risk, and the marginal contribution of stock options is, in fact, small.

In view of the important role that leverage plays for incentives, we then explore the relevance of asset risk-taking incentives in a sample of U.S. financial institutions. We are particularly interested in the years before the 2007/08 financial crisis.

Here, too, we document several notable facts, which confirm in the data what the previous analysis had suggested for a hypothetical CEO. *First*, for many CEOs in the sample, incentives to take asset risk emanating from stock-holdings are substantial. In some contrast, relying on results from Guay (1999) for an average of firms over a range of industries, the existing literature argues that the incentives, due to stock holdings, to take equity risk are negligible. By setting Equity Volatility Vega from stock to zero, the literature effectively assumes that risk-taking incentives are due *only* to stock options. Practitioners also tend to perceive stock options as the main driver of risk-taking behavior by CEOs. Our theoretical and empirical findings qualify this view, and they have both practical implications for boards of directors designing managerial compensation systems and implications for research on risk-taking.

*Second*, a puzzling fact in much of the literature is that the observed Equity Incentive Ratio is often fairly small – perhaps too small to offer an economic justification for a CEO to engage in significant risk-taking. For asset incentives, we obtain quite different results. In our sample, for the main parameter choice, the average CEO has an Asset Volatility Vega of around US$3.5 million and an Asset Delta of around US$6 million dollars. The Asset Incentive Ratio is, at the mean, about 0.50, around 30-50% larger than the Equity Incentive Ratio. For other reasonable parameter values, this difference can be much larger.

*Third*, there is not only a difference in levels between asset and equity incentives, but also in the cross-sectional pattern. The correlation between the Asset Incentive Ratio and the Equity Incentive Ratio is on average only about 0.40 in our sample. Thus, incorporating cross-sectional variation in leverage into the calculation of incentives and recognizing that risk-taking incentives emanate also from stocks indeed yields a new perspective on incentives.

Section II then provides a topical example of the potential importance of considering managerial asset incentives when aiming to explain asset risk-taking, namely the 2007/08 crisis. This crisis essentially exposed the downside of the asset risk that banks undertook in prior years. As such, write-downs in 2007/08 form a natural indication of the degree of asset risk-taking by financial institutions in years prior to the crisis.[2]

We find that, by and large, banks with higher Asset Incentive Ratios in the years 2003-05 (and 2006 when controlling for governance) had higher write-downs (both in absolute terms and scaled by total assets). In other words, incentives help explain variation in asset risk-taking by financial institutions. Interestingly, the Equity Incentive Ratio does not provide much explanatory power in our regressions, consistent with the intuition that to explain asset risk-taking it is important to use

[2] In the data section, we discuss issues such as the discretion companies have in setting the level of write-downs, as well as the benefits and challenges of using alternative measures of risk-taking.

asset incentives.

### I.1. Related literature

Our paper contributes to two strands of the literature. *First*, the general literature on managerial compensation schemes and their consequences, which is too large to review in detail, mostly deals with incentives of the CEO to increase the share price and to take equity risk (see, for example, Coles, Daniel, and Naveen (2006), Guay (1999) and Knopf, Nam, and Thornton (2002)). The main contribution of this paper is to introduce and shed light on the relevance of managerial incentives to take *asset* risk and increase the firm value. Although asset substitution has been a central theme of corporate finance since Jensen and Meckling (1976), we do not know of other empirical studies of asset incentives.

*Second*, this paper also makes a specific contribution to the financial crisis literature because our approach and, consequently, our results differ from other papers.[3] The related papers in this literature can be organized in terms of the dependent variable (risk-taking before the crisis or performance during the crisis) and the central explanatory variable (incentives or governance), as shown in the following matrix.

The work listed in the four quadrants is as follows: *First*, some papers consider risk-taking and governance: Cheng, Hong, and Scheinkman (2012) document a positive association between total compensation and risk-exposure of financial institutions; other work focuses on corporate governance and specific measures of risk-taking, such as risky mortgage-backed securities involvement (Ellul and Yerramilli, 2013). *Second*, Adams (2012) considers the quality of governance in U.S. banks, while Erkens, Hung, and Matos (2012) primarily consider the relation between various measures of performance and corporate governance in a global sample. (They also provide results regarding risk-taking.) *Third*, Fahlenbrach and Stulz (2011) study the correlation of performance measures,

[3]For studies on risk-taking and governance in banks generally see, for example, Laeven and Levine (2009) and Saunders, Strock, and Travlos (1990). Faulkender, Kadyrzhanova, Prabhala, and Senbet (2010) survey some of the financial crisis literature.

stock returns and returns on assets, with equity incentives.

**Selection of related empirical papers in the financial crisis literature**

| | **Risk-Taking** | **Performance** |
|---|---|---|
| **Incentives** | DeYoung, Peng, and Yan (2012)<br>***This paper*** | Fahlenbrach and Stulz (2011) |
| **Governance** | Cheng, Hong, and Scheinkman (2011)<br>Ellul and Yerramilli (2009) | Erkens, Hung und Matos (2009)<br>Adams (2012) |

*Fourth*, DeYoung, Peng, and Yan (2012) consider the relationship between incentives and risk-taking. They provide a rich set of evidence. Of particular interest, in relation to our analysis, are their results on asset risk-taking. They find that both Equity Volatility Vega *and* Equity Delta were positively associated with investments in MBS securities in the years before the crisis. As they note, this finding is puzzling. Theoretical considerations and previous empirical evidence suggest that Vega and Delta should be associated with risk-taking with opposite signs (e.g., Knopf, Nam, and Thornton (2002)).[4] Our analysis provides empirical support for this idea in the context of the 2007/08 financial crisis.

## II. Managerial Asset Incentives

We consider a CEO who receives equity-based compensation, consisting of stock and/or stock options. To measure managerial incentives to take asset risks, we appeal to the idea, dating back to Merton (1974), that equity can be viewed as a contingent claim on the firm value. Using this framework, we calculate *asset incentives*, *Asset Volatility Vega* – capturing the incentives to increase the standard deviation of firm value – and *Asset Delta* – capturing the incentives to increase firm

[4]In an earlier working paper version, De Young, Peng and Yan looked at an even broader range of asset risk variables, for example, bank investments in commercial real estate and mortgages. Like MBS investments, these other variables are arguably related to our summary measure, write-downs. However, they obtained mixed results with these variables. For equity risk-taking measures, systematic equity risk and idiosyncratic equity risk, DeYoung, Peng, and Yan (2012) find a positive relation with Equity Volatility Vega and a negative relation with Equity Delta, consistent with theoretical predictions.

value. Guay (1999) also relies on the contingent claims idea to calculate, for a sample of firms including both financial and non-financial companies, the Equity Volatility Vega of stock. However, Guay (1999) then goes on to calculate Equity Volatility Vega from stock options in the usual way, relying on the Black-Scholes formula, treating equity as the primitive, and then adds to this the Equity Volatility Vega from stocks obtained by relevering his Asset Volatility Vega of stocks. We instead consider stock options as compound options (Geske, 1979). The idea of considering stock options as compound options on the firm value is new to the literature on risk-taking, but a natural step in order to operate within a coherent framework.

Subsection II.1 sets up the model framework. Subsection II.2 derives Asset Delta and Asset Volatility Vega. Subsection II.3 introduces Equity Delta and Equity Volatility Vega. Subsection II.4 explains the differences of asset incentives and equity incentives for the case of a single stock or stock option. Subsection II.5 investigates the incentives emanating from a portfolio of stocks and stock options. Subsection II.6 provides a quantitative analysis of managerial incentives to take asset risk in a cross section of financial institutions.

### II.1. Model

We follow Merton (1974) and assume the firm value, $V$, follows a geometric Brownian motion

$$\frac{dV_t}{V_t} = \mu_V dt + \sigma_V dW_t \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion under the historical measure. The parameters $\mu_V$ and $\sigma_V$ are assumed constant. We also assume that there exists a bank account which yields a constant interest rate $r$. Considering equity as a call option on the firm value, stock can be valued according to the Black-Scholes formula[5],

$$\mathrm{BS}_0 := S_0 = V_0 N(d_1) - De^{-rT_D} N(d_2) \tag{2}$$

where $d_1 = \frac{\ln(V_0/D)+(r+\sigma_V^2/2)T_D}{\sigma_V\sqrt{T_D}}$ and $d_2 = d_1 - \sigma_V\sqrt{T_D}$.

Next, we recognize that stock options can be viewed as compound options on the underlying firm value $V$. This idea is novel to the executive incentives literature. Geske (1979) shows that the price of a compound call (CC) option is given by[6]

$$\begin{aligned} \mathrm{CC}(V, D, r, T_{CC}, T_D, \sigma_V, K) &= VN_2(h + \sigma_V\sqrt{T_{CC}}, k + \sigma_V\sqrt{T_D}; \sqrt{T_{CC}/T_D}) \\ &- De^{-rT_D} N_2(h, k; \sqrt{T_{CC}/T_D}) - Ke^{-rT_{CC}} N_1(h), \end{aligned} \tag{3}$$

where $N_2(\cdot)$ represents the bivariate cumulative normal distribution function, $N_1(\cdot)$ represents the standard normal cumulative distribution function, $T_{CC}$ denotes the expiration of the stock option and $T_D$ denotes the maturity of the debt.[7] The remaining terms are

$$h = \frac{\ln(V/\bar{V}) + (r - \frac{1}{2}\sigma_V^2)T_{CC}}{\sigma_V\sqrt{T_{CC}}}, \quad \text{and} \quad k = \frac{\ln(V/D) + (r - \frac{1}{2}\sigma_V^2)T_D}{\sigma_V\sqrt{T_D}} \tag{4}$$

and $\bar{V}$ is the value of $V$ which is the implicit solution to the equation

$$VN_1(\bar{k}(V) + \sigma_V\sqrt{T_D - T_{CC}}) - De^{-r(T_D - T_{CC})} N_1(\bar{k}(V)) - K = 0, \tag{5}$$

[5]We exclude dividends for simplicity. From option pricing theory there is no reason to expect that omitting this component will result in substantially different outcomes.

[6]From here on, we omit the time subscripts for notational simplicity.

[7]Thus, the model requires that $T_D > T_{CC}$. We come back to the choice of appropriate debt and stock option maturities in our application below.

where

$$\bar{k}(V) = \frac{\ln(V/D) + (r - \frac{1}{2}\sigma_V^2)(T_D - T_{CC})}{\sigma_V \sqrt{T_D - T_{CC}}} \tag{6}$$

and where $K$ denotes the strike price of the option and $D$ denotes the face value of debt per share, so that $\bar{V}$ denotes the firm value where the option is just at the money at time $T_{CC}$.

## II.2. Asset Incentives

We now compute sensitivities of a single stock and a single stock option with respect to firm value parameters. *Asset Delta* is the first derivative of the stock and stock option price, respectively, with respect to a one percent change in the firm value, $V$. *Asset Volatility Vega* is the first derivative of the stock and stock option price, respectively, with respect to a 0.01 change in the underlying asset return volatility, $\sigma_V$.

*First*, the sensitivity of a stock with respect to a change in firm value is

$$\begin{aligned}\text{Asset Delta from stocks} &= \frac{\partial \text{BS}_0(V, D, r, T_D, \sigma_V)}{\partial V} \cdot (V/100) \\ &= N(d_1(V, D, r, T_D, \sigma_V)) \cdot (V/100). \end{aligned} \tag{7}$$

Note that Asset Delta from stocks is not equal to one.

*Second*, by relying on the formula for a compound call option,

$$\begin{aligned}\text{Asset Delta from stock options} &= \frac{\partial \text{CC}(V, D, r, T_{CC}, T_D, \sigma_V, K)}{\partial V} \cdot (V/100) \\ &= N_2(h + \sigma_V \sqrt{T_{CC}}, k + \sigma_V \sqrt{T_D}, \sqrt{T_{CC}/T_D}) \cdot (V/100). \end{aligned} \tag{8}$$

Note that the derivative of the compound formula with respect to the underlying asset value, $V$, converges to the Delta in the Black-Scholes model as the face value of debt equals zero.

*Third*, we compute the incentive, coming from the CEO's stock holdings, to increase the firm value return volatility. Relying on the Black-Scholes formula, we obtain

$$\begin{aligned}\text{Asset Volatility Vega from stocks} &= \frac{\partial \text{BS}_0(V, D, r, T_D, \sigma_V)}{\partial \sigma_V} \cdot (1/100), \\ &= \varphi(d_1(V, D, r, T_D, \sigma_V)) V \sqrt{T_D} \cdot (1/100), \qquad (9)\end{aligned}$$

where $\varphi$ denotes the standard normal probability density.

*Fourth*, we turn to the incentive, coming from the CEO's stock option holdings, to increase the firm value return volatility. For this sensitivity, we derive the result in Proposition 1.

***Proposition 1.*** *The Asset Volatility Vega of a stock option in the compound option pricing model is given by*

$$\textit{Asset Volatility Vega from stock options} = \frac{\partial CC(V, D, r, T_{CC}, T_D, \sigma_V, K)}{\partial \sigma_V} \cdot (1/100) \qquad (10)$$

*where*

$$\begin{aligned}\frac{\partial CC(V, D, r, T_{CC}, T_D, \sigma_V, K)}{\partial \sigma_V} &= V \varphi(h + \sigma_V \sqrt{T_{CC}}) \sqrt{T_{CC}} N_1 \left( \bar{k}(\bar{V}) + \sigma_V \sqrt{T_D - T_{CC}} \right) \\ &+ D e^{-rT_D} \varphi(k) \sqrt{T_D} N_1 \left( \frac{h - \sqrt{\frac{T_{CC}}{T_D}} k}{\sqrt{1 - \frac{T_{CC}}{T_D}}} \right) \qquad (11)\end{aligned}$$

*and the parameters are defined in Section II.1.*

***Proof*** See Appendix A.

Contrary to the result presented in Geske (1979), which is shown in Appendix A for completeness, our formula for the derivative of the compound option value with respect to the asset return volatility converges to the Black-Scholes Vega as the debt converges to zero. Indeed, this appears to be the intuitive benchmark result. In the formula of Geske (1979), vega goes to zero for zero debt. More importantly, there are often substantial differences in the magnitude of the calculated risk-taking incentives using both approaches even for non-zero debt. This is illustrated in Figure 1. The figure considers a single stock option. We plot the Asset Volatility Vega using the formula in Geske (1979) and the formula given in Proposition 1. Figure 1 also shows a discrete difference approximation using the compound option pricing formula derived by Geske (1979).[8] Notably, this difference approximation agrees with our analytical formula for the Asset Volatility Vega given in Proposition 1.

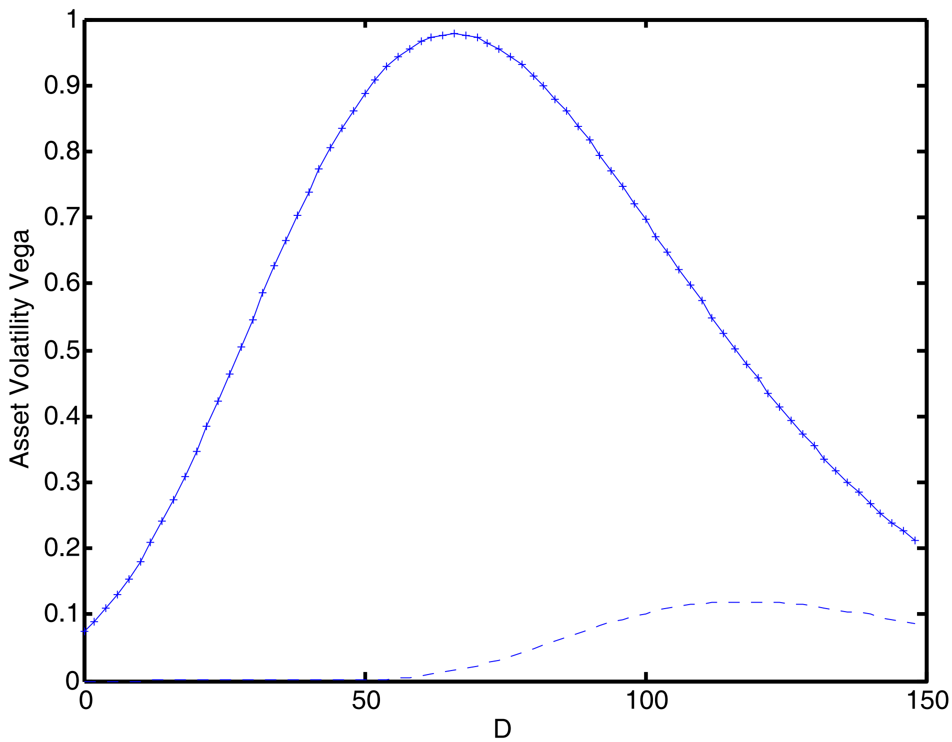


**Figure 1.** Asset Volatility Vega as a function of debt.

This graph plots Asset Volatility Vega for varying levels of the face value of debt, $D$, computed using three different approaches: The Geske (1979) approach (dotted line), our analytical result derived in Proposition 1 ("+") and the difference approximation (solid line). See Section II.2 for details. The parameters are $V = 100$, $K = 50$, $r = 0.04$, $T_D = 10$, $T_{CC} = 6$ and $\sigma_V = 0.10$. $\Delta = 10^{-11}$ is used to calculate the difference approximation.

For the chosen parameters and a face value of debt of 85 (which implies leverage of around 55%),

[8] That is, we approximate the derivative of the compound call option price with respect to $\sigma_V$ with a first difference of the compound option pricing formula with respect to $\sigma_V$. For this, we consider a sufficiently small change in $\sigma_V$ in order to approximate the true derivative with a high precision.

a single compound call option on the stock is worth US$ 11.5. For this case, we observe an Asset Volatility Vega (from a single stock option) of around US$ 0.94 based on Proposition 1, whereas the equivalent Asset Volatility Vega based on the formula presented in Geske (1979) equals US$ 0.12.

Naturally, at very high leverage levels, when the compound option is deep out of the money, increases in the asset volatility induce small value changes in the option value, resulting in the same hump-shape form of Asset Volatility Vega of a stock option with respect to the debt level that arises also for a stock option with respect to the stock option strike price. Asset Volatility Vega is monotonically increasing in the underlying asset volatility; see Figure A-1 in Appendix A.

### II.3. Equity Incentives

An alternative way to quantify managerial incentives is to consider the sensitivities of CEO wealth to the stock price level and stock return volatility, respectively. *Equity Delta* is the sensitivity of a stock or stock option with respect to a one percent change in the company's underlying stock price. *Equity Volatility Vega* is the change in the dollar value of a stock or stock option in response to a one percentage point change in stock return volatility based on the Black-Scholes option pricing formula. Following Core and Guay (2002) and most of the existing literature, we assume that the Equity Volatility Vega from shares of stock equals zero. See Appendix B for details.

### II.4. Comparing Asset Incentives and Equity Incentives

Asset incentives depend explicitly on the level of leverage, consistent with the central insight of Jensen and Meckling (1976) that asset substitution incentives increase with leverage. By contrast, leverage does not enter explicitly into the formulas determining equity incentives shown in Appendix B. Thus, Equity Delta is unity, but Asset Delta is less than unity. Conversely, Asset Volatility Vega is greater than Equity Volatility Vega. *For a given firm* (at a given leverage), the latter result is merely a reflection of the fact that equity volatility is determined by asset volatility, multiplied by the elasticity of the equity value with respect to the asset value ($\sigma_S = \sigma_V (dS/S)/(dV/V)$). This elasticity is greater than one. Importantly, it is a (non-linear) function of leverage; thus, it varies across firms. *In the cross section of firms* with varying degrees of leverage this fact induces a critical difference between asset incentives and equity incentives.

Intuitively, when working with equity incentives, an increase in equity volatility by 0.01 comes about from different increases of asset risk for each CEO because firms differ in leverage. Thus, the cross section of incentives to take equity risk yields limited insight into the cross section of CEO incentives to take asset risk. Considering asset incentives, instead, "normalizes" the incentive measure in the sense that an increase of asset volatility by 0.01 means the same for each CEO.[9] The following two subsections develop this intuition in more detail and explore its quantitative importance in practice.

### II.5. Incentives From a Portfolio of Stock and Stock Options

Consider a board that provides a CEO with an equity-based compensation package. The package should have some given value (say, US$ 5 million, though this value is irrelevant for the analysis that follows). The total Vegas and Deltas are obtained by multiplying the single stock and single stock option Vegas and Deltas by the numbers of stocks and stock options conveyed to the CEO. What are the implied managerial asset substitution incentives of different combinations of stock and stock options, holding the overall value of the compensation package constant?

To analyze this issue in a condensed fashion, we begin by noting that, clearly, asset risk-taking incentives are higher if Asset Volatility Vega is higher. In addition, a risk-averse CEO wishes to avoid fluctuations in the asset value, and this desire is more pronounced the more he participates in any upward or downward movement of firm value. Therefore, asset risk-taking incentives are lower if Asset Delta is higher. In the spirit of Dittmann and Yu (2011), we therefore define the

[9]Of course, it may be more difficult for some CEOs to achieve such an increase than for others, but that is not a matter of incentives, but a matter of cost of effort.

*Asset Incentive Ratio* as the ratio of Asset Volatility Vega and Asset Delta.[10] This provides a useful summary measure of incentives for our purposes. Similarly, the *Equity Incentive Ratio* is the ratio of Equity Volatility Vega and Equity Delta.

Figure 2 shows three lines each for the Asset and Equity Incentive Ratio, using different levels of leverage. Several properties of the incentive ratios are noteworthy.

*First*, the intercept of the Asset Incentive Ratio line is above zero – consistent with the fact that the asset view allows for risk-taking incentives to emanate also from stock holdings. Naturally, the intercept is higher for higher leverage.

*Second*, the Asset Incentive Ratio is larger than the Equity Incentive Ratio. In other words, asset risk-taking incentives can be large relative to incentives to increase firm value, even when incentives to take equity risk are dwarfed by incentives to increase the stock price.

*Third*, over the whole range of portfolio combinations, as leverage increases, asset substitution incentives for managers become more pronounced and the difference between the Asset Incentive Ratio and the Equity Incentive Ratio becomes bigger. (The same is naturally also true for higher asset volatility; this result is not shown in the graph.) For high leverage, the Asset Incentive Ratio is easily greater than unity, while this only occurs in extreme cases for the Equity Incentive Ratio.

*Fourth*, when stock options make up a larger fraction of the portfolio of the CEO, the overall risk-taking incentives, measured by the two incentive ratios, increase. But the role of stock options varies between asset incentives and equity incentives. For example, starting at 80% stock and 20% stock options and going to 20% stock and 80% stock options, the Equity Incentive Ratio goes up by a factor of three. By contrast, for the same change in the composition of the pay package the Asset Incentive Ratio only approximately doubles (in the case of leverage of 40% and 55%) or increases by only about 40% (in the case of leverage of 70%). This result can be explained by recognizing that when leverage is higher, stock itself already incorporates a significant optionality (reflected in the

[10] Lambert, Larcker, and Verrecchia (1991), Ingersoll (2006), and Carpenter (2000) also argue that a risk-averse and under-diversified manager may adopt risk-reducing policy choices when compensation exhibits high pay-performance sensitivity.

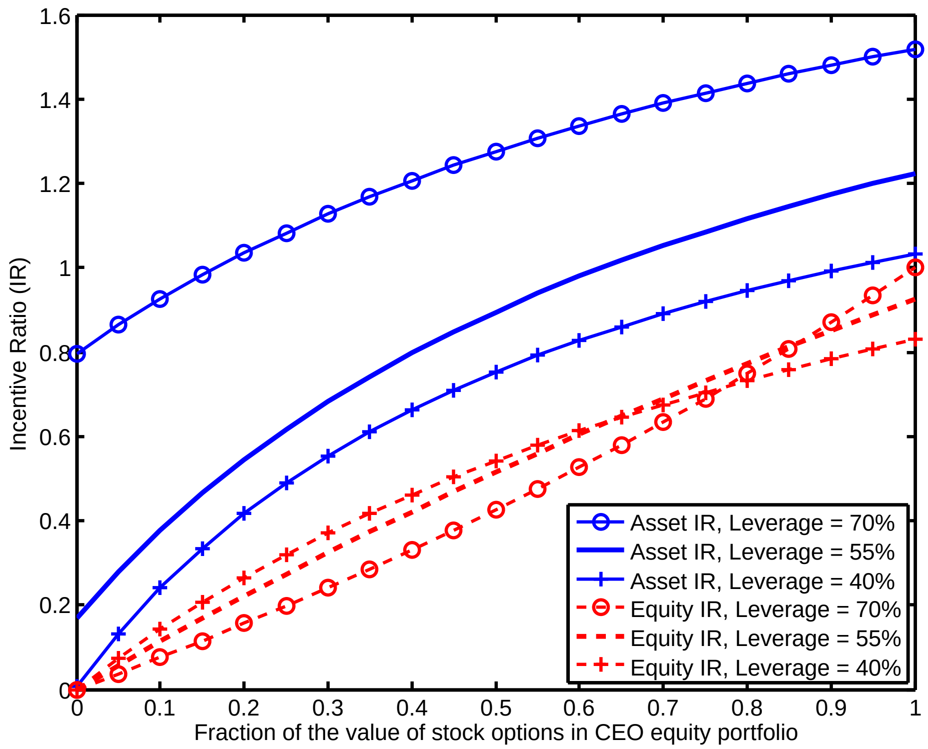


**Figure 2.** Incentive ratios.

This graph plots the two incentive ratios (IR), the Asset Incentive Ratio (solid lines) and the Equity Incentive Ratio (dotted lines), for different combinations of stocks and stock options, for a given value of the total equity-based compensation package. For the Asset Incentive Ratio, sensitivities of stocks and stock options are calculated according to the model described in Section II.2, using in particular the Merton (1974) model and the compound option framework for stock options (Geske, 1979), but using our analytical result for Asset Volatility Vega of a stock option (see Proposition 1). The parameters for the Asset Incentive Ratio are $V = 100$, $r = 0.04$, $T_D = 10$, $T_{CC} = 6$ and $\sigma_V = 0.10$. For the Equity Incentive Ratio, sensitivities are calculated as described in Section II.3, using the Black-Scholes model, also using 6 years as the stock option maturity. The face value of debt is varied to capture different degrees of leverage. Variation in leverage implies variations in the value of equity according to the Merton (1974) model. For both incentive ratios, we choose the strike price for the stock options to be equal to the stock price implied from the Merton (1974) model at each point, i.e., stock options are granted at-the-money. The stock return volatility is numerically determined through the portfolio relationship described in Appendix C.

higher intercept), so that the extra optionality introduced by the compound option adds relatively little to the overall asset risk-taking incentives.[11]

## II.6. The Cross Section of CEO Incentives in Financial Institutions

We have seen that, consistent with intuition, asset substitution incentives of CEOs are likely to become particularly pronounced at high leverage. We now study a group of firms for which leverage is a central characteristic: financial institutions. Indeed, understanding incentives of managers in the financial services industry is of particular interest in the light of recent events.

The sample in which we study this issue includes depository institutions, non-depository credit institutions, and investment banks and some brokerage firms. We refer to all companies in our sample as *financial institutions*.[12] We require a company to have compensation data available in the ExecuComp database. We focus on the years 2003 to 2006 because, as we explain further below, these years are most likely to contain information regarding the incentives of managers related to asset risk-taking that became relevant in the crisis. We also require that the company is alive at the beginning of the third quarter of 2007. Companies which did not have a stock price observation in CRSP for July 2007 are deemed inactive and excluded from our analysis. However, companies remaining on this list are allowed to subsequently default, be taken over by or merged with another company during the crisis period.[13]

[11] One can also verify that, when leverage is higher, the maximum difference between the Asset Incentive Ratio and the Equity Incentive Ratio occurs at a lower fraction of stock options.

[12] The sample choice is motivated by the empirical analysis of asset risk-taking by financial institutions that we conduct later. Table C-1 in Appendix C displays the SIC codes we consider in the analysis.

[13] Some companies were too small to remain in the ExecuComp database for all years 2003 to 2006 but were still alive going into July 2007. Their compensation information is included into the analysis for the year(s) when they are covered in the ExecuComp database.

### II.6.1. Inputs

Information about executive compensation pay packages is available annually for U.S. entities in Standard & Poor's ExecuComp database. We focus our analysis on the CEO.[14] Consistent with existing literature, we define the inputs for our calculations as follows: (1) The stock options exercise prices and (2) maturities are taken from ExecuComp to the extent these are available. If the exercise price for granted stock options is not available, we assume they were granted at-the-money. To obtain the stock option maturity for missing grant dates, we follow Guay (1999) and calculate the maturity of the stock options by assuming that the stock options were granted on July 1 of the year in which the stock options were granted. (3) We use the fiscal year end closing price of the given year as the current stock price. (4) The stock return volatility is calculated (from CRSP data) as the annualized standard deviation of daily log-returns over the past three years by assuming 250 trading days in the year. (5) We use the U.S. Treasury yields obtained from the Fed's webpage as proxies for the risk-free interest rate. (6) The annual cash dividend paid by the company over the fiscal year end closing price is used to calculate the dividend yield. This information is also from CRSP.

For the asset incentives, we follow Guay (1999) in the calculations of the implied firm value and other parameters needed for the calculation of incentives. In particular, (7) the firm value (return) volatility is determined through a portfolio relationship with stock volatility and debt volatility (see Appendix C for details). For the standard deviation of debt returns, we use the annualized standard deviation calculated on monthly (log) returns using the Merrill Lynch Bank of America corporate financial bond index using a five year period.[15] As in Guay (1999), (8) the strike price for equity,

[14] Murphy (2011) argues that incentives of traders were more important than those of CEOs. Data on below-management-level compensation structures are not broadly available, unfortunately. For years with missing CEO information and where the dates at which the CEO assumed office is prior to the particular year, we classify the executive as the CEO accordingly. If the CEO is not recorded and the necessary information is not accessible from the SEC Edgar database, we do not include the respective firm in the analysis.

[15] We believe that this index fits our purposes better than the general Merrill Lynch Bank of America corporate bond index that matches the S&P senior debt rating which Guay (1999) uses. With the index used in Guay (1999), we obtain stronger results both in terms of the size of risk-taking incentives and the statistical significance in our write-down regressions. Finally, the approach used in Gropp and Heider (2010), just delevering stock return volatility, yields very similar overall results, too.

seen as an option on firm value, is the book value of debt per share. (9) Guay (1999) assumes for financial firms that they have a single debt obligation with time to maturity equal to 7.5 years (as for most financial firms maturity data is unavailable). We use the same baseline assumption.[16] Together with the observation that the stock can be considered as a call option on the firm value, one can finally back out (10) an implied firm value per share. Table C-2 shows descriptive statistics for all the relevant variables.

Delta and Vega for previously granted options (i.e., exercisable and un-exercisable options) and current year granted options are then multiplied by the amount of options held by the CEO in each of these dimensions to form the final Vega and Delta quantities.

### II.6.2. Asset and Equity Incentives in Financial Institutions

As a benchmark, we begin by analyzing the results for equity incentives. As seen in Table I, the average Equity Volatility Vega implies that a one percentage point increase in the company's stock price volatility is associated with an increase of around US$301,000 in CEO wealth. This number is comparable in size with that of other studies on risk-taking incentives in banks. While Equity Volatility Vega in financial institutions is about double the size of Equity Volatility Vega in the Coles, Daniel, and Naveen (2006) sample of industrial firms, it still seems modest, in particular compared to Equity Delta. A 1% increase in the company's stock price results in an increase of around US$1,131,000 in the CEO's wealth on average. The Equity Incentive Ratio, defined as the ratio of Equity Volatility Vega and Equity Delta, is 0.30 on average.

However, it would be wrong to conclude, on the basis of evidence from equity incentives, that

[16] Core deposits (in addition to long-term debt as funding channels in financial institutions) have no explicit maturity and are often referred to as non-maturity debt. As pointed out by Sheehan (2013) such deposits often remain within the financial institution for significant periods of time, often longer than 10 years. Thus, 7.5 years is a reasonable approximation. Note that the debt maturity needs to be longer than the stock option maturity in order for the compound option approach to apply. In practice, stock option maturities typically range from 3 to 10 years. In cases where the stock option maturity is longer than 7.5 years, we set the maturity of debt equal to the stock option maturity plus two years. That is, for a stock option with a 10 year maturity, we effectively assume that the debt maturity is 12 years. However, consistent with Guay's (1999) assessment of the sensitivity of his results for Asset Volatility Vega from stocks to assumptions about debt maturity, we find that for Asset Volatility Vega from options, the results do not appear sensitive to how we adjust the debt maturity either.

the wealth-driven incentives of CEOs to engage in risky activities, such as investing in sub-prime products, is small. The Asset Incentive Ratio, that is, the ratio of Asset Volatility Vega to Asset Delta, equals 0.44 at the median and is, as such, about 50% larger than the Equity Incentive Ratio. Interestingly, the correlation of Asset Volatility Vega with Equity Volatility Vega is far from perfect, at 0.5 to 0.7 across the years. And the correlation between the Asset Incentive Ratio with the Equity Incentive Ratio is only around 0.4. This set of results reflects the fact that, as explained in Subsection II.4, asset incentives explicitly take into account leverage.

Moreover, Table I also shows that, as expected, a large part of asset risk-taking incentives continues to come from stock options, but it is clear that stock holdings can also imply significant asset risk-taking incentives. For the median CEO, asset risk-taking incentives due to options are large compared to those due to stock, but for the mean CEO, (only) about half of the total incentives to increase the asset return volatility are due to options. Table II confirms that the results holds across the years under consideration.

At higher asset volatilities, these effects are even more pronounced. This is important because it is often argued that in recessions asset volatilities increase, suggesting particularly powerful asset risk-taking incentives of managers in bad times, relative to incentives to increase firm value. For instance, using debt volatilities proposed by Guay (1999) in his analysis (which covers both financials and industrial firms) yields somewhat higher asset volatilities and, consequently, ratios of Asset Vega Volatility to Asset Delta that are easily twice or three times as large as the ratios of Equity Volatility Vega and Equity Delta. This second version is shown as "V2" in Table I.

In sum, considering incentives to take asset risk yields a novel picture of managerial risk-taking incentives and may, therefore, contribute to an enhanced understand of the relationship of incentives with risk-taking. We explore this potential in the next section.

**Table I.** Summary statistics I.

Summary statistics of company specific variables, write-downs and the risk-taking incentives averaged over the years 2003-2006. The variables are described in Table III. All variables are winsorized at the 2nd and 98th percentile on an annual basis. The Asset Incentive Ratio and the Equity Incentive Ratio represent Asset Volatility Vega divided by Asset Delta and Equity Volatility Vega divided by Equity Delta, respectively. For details on parameter choices, see the text. Asset Incentive Ratio (V2) denotes a second version of the Asset Incentive Ratio. In that version, we use Guay's (1999) assumptions for the standard deviation of debt returns, which implies somewhat higher asset volatilities than in our main case. The regressions in Tables 5 and 6 use the main version of the asset incentives. The term "q" denotes the quantile, i.e., 10q refers to the 10th quantile in the empirical distribution of the particular variable. All monetary values are expressed in 2008 dollars.

| | Mean | Std. Dev. | 10q | 25q | median | 75q | 90q |
|---|---|---|---|---|---|---|---|
| *Write-downs* | | | | | | | |
| Write-downs (USD mill.) | 6644.3 | 16653.7 | 18.0 | 101.9 | 468.5 | 2981.1 | 21736.5 |
| Write-downs scaled by total assets | 0.055 | 0.082 | 0.003 | 0.012 | 0.029 | 0.066 | 0.140 |
| *CEO incentives* | | | | | | | |
| Equity Volatility Vega (1000 USD) | 301.5 | 423.0 | 5.3 | 27.9 | 103.3 | 403.6 | 897.1 |
| Equity Delta (1000 USD) | 1130.9 | 1638.5 | 42.3 | 171.9 | 549.0 | 1264.1 | 2955.9 |
| Equity Incentive Ratio | 0.35 | 0.30 | 0.03 | 0.10 | 0.30 | 0.52 | 0.74 |
| Asset Volatility Vega (1000 USD) | 3459.5 | 6538.0 | 39.7 | 161.4 | 930.9 | 3039.2 | 10660.5 |
| due to options (1000 USD) | 2225.4 | 3609.4 | 39.8 | 178.4 | 831.8 | 2192.1 | 6368.1 |
| due to stocks (1000 USD) | 1522.6 | 4604.0 | 5.3 | 25.7 | 129.5 | 691.3 | 2664.1 |
| Asset Delta (1000 USD) | 6054.5 | 9627.8 | 203.5 | 651.5 | 2446.0 | 6742.3 | 17048.7 |
| due to options (1000 USD) | 3486.2 | 4817.8 | 126.9 | 368.0 | 1609.2 | 4099.3 | 11394.3 |
| due to stocks (1000 USD) | 2930.4 | 6307.3 | 86.6 | 252.5 | 803.2 | 2304.3 | 6822.8 |
| Asset Incentive Ratio | 0.49 | 0.34 | 0.10 | 0.23 | 0.44 | 0.67 | 0.93 |
| Asset Incentive Ratio (V2) | 0.90 | 0.50 | 0.27 | 0.57 | 0.89 | 1.19 | 1.57 |
| *Firm characteristics* | | | | | | | |
| Market Cap. (USD mill.) | 15990.2 | 29386.6 | 693.1 | 1080.5 | 2811.2 | 15611.8 | 55234.6 |
| Total Assets (USD mill.) | 109778.4 | 240345.5 | 2535.8 | 5082.5 | 13798.2 | 79723.0 | 350432.6 |
| Book-to-Market Ratio (%) | 0.49 | 0.18 | 0.25 | 0.37 | 0.49 | 0.60 | 0.73 |
| Book Leverage (%) | 0.90 | 0.07 | 0.86 | 0.90 | 0.91 | 0.93 | 0.94 |
| Market Leverage (%) | 0.80 | 0.11 | 0.72 | 0.78 | 0.82 | 0.86 | 0.89 |
| *Governance* | | | | | | | |
| Percentage independent directors | 72.9 | 12.9 | 55.6 | 66.7 | 75.0 | 82.4 | 87.5 |
| Tenure | 7.7 | 5.8 | 2.0 | 3.0 | 6.0 | 11.0 | 16.0 |
| Governance index | 9.8 | 2.9 | 6.0 | 8.0 | 10.0 | 12.0 | 14.0 |

**Table II.** Summary statistics II.

Summary statistics of Asset Volatility Vega and Asset Delta divided into incentives coming from stock holdings and stock options, respectively, across the years 2003-2006 for all financial institutions in our sample. The Asset Incentive Ratio and the Equity Incentive Ratio represent Asset Volatility Vega divided by Asset Delta and Equity Volatility Vega divided by Equity Delta, respectively. The variables are winsorized at the 2nd and 98th percentile on an annual basis. The term "q" denotes the quantile, i.e., 10q refers to the 10th quantile in the empirical distribution of the respective variable. All monetary values are denominated in 1000 USD and expressed in year 2008 dollars.

| Year | Variable | Mean | Std. Dev. | 10q | 25q | median | 75q | 90q |
|---|---|---|---|---|---|---|---|---|
| 2003 | Asset Volatility Vega | | | | | | | |
| | due to stocks | 1526.5 | 3944.8 | 8.8 | 37.2 | 171.5 | 777.8 | 3682.5 |
| | due to options | 2718.9 | 4195.1 | 48.0 | 218.8 | 1063.4 | 3128.7 | 8908.9 |
| | Asset Delta | | | | | | | |
| | due to stocks | 2566.2 | 5046.2 | 61.1 | 226.7 | 769.4 | 2148.5 | 5864.3 |
| | due to options | 3555.7 | 4662.5 | 135.7 | 380.1 | 1888.5 | 4546.6 | 10730.2 |
| | Asset Incentive Ratio | 0.63 | 0.37 | 0.20 | 0.33 | 0.61 | 0.88 | 1.09 |
| | Equity Incentive Ratio | 0.37 | 0.25 | 0.05 | 0.16 | 0.36 | 0.52 | 0.77 |
| 2004 | Asset Volatility Vega | | | | | | | |
| | due to stocks | 1754.6 | 5064.1 | 8.0 | 26.8 | 145.5 | 751.0 | 2808.3 |
| | due to options | 2461.9 | 3929.3 | 53.4 | 197.7 | 938.6 | 2511.4 | 9498.3 |
| | Asset Delta | | | | | | | |
| | due to stocks | 2867.8 | 5939.3 | 91.5 | 237.8 | 848.3 | 2468.1 | 6682.0 |
| | due to options | 3686.7 | 5212.6 | 136.0 | 422.7 | 1745.5 | 4429.1 | 10661.7 |
| | Asset Incentive Ratio | 0.55 | 0.32 | 0.14 | 0.32 | 0.52 | 0.74 | 0.96 |
| | Equity Incentive Ratio | 0.36 | 0.24 | 0.05 | 0.14 | 0.31 | 0.57 | 0.70 |
| 2005 | Asset Volatility Vega | | | | | | | |
| | due to stocks | 2045.5 | 6465.7 | 6.4 | 28.7 | 138.3 | 1048.5 | 2391.4 |
| | due to options | 2299.5 | 3526.1 | 91.7 | 198.3 | 906.4 | 2602.8 | 8460.8 |
| | Asset Delta | | | | | | | |
| | due to stocks | 3687.9 | 9121.7 | 86.3 | 370.2 | 834.1 | 2411.4 | 6297.8 |
| | due to options | 3846.3 | 5181.1 | 135.9 | 398.4 | 1724.0 | 4321.6 | 12509.7 |
| | Asset Incentive Ratio | 0.51 | 0.32 | 0.12 | 0.28 | 0.50 | 0.67 | 0.95 |
| | Equity Incentive Ratio | 0.39 | 0.30 | 0.04 | 0.13 | 0.33 | 0.59 | 0.82 |
| 2006 | Asset Volatility Vega | | | | | | | |
| | due to stocks | 879.8 | 2112.9 | 2.3 | 9.7 | 78.8 | 448.7 | 2029.6 |
| | due to options | 1557.9 | 2728.1 | 25.5 | 81.0 | 410.4 | 1783.0 | 3983.3 |
| | Asset Delta | | | | | | | |
| | due to stocks | 2633.9 | 4307.1 | 86.6 | 219.4 | 792.5 | 2497.2 | 7519.0 |
| | due to options | 2954.8 | 4278.2 | 75.8 | 248.6 | 1058.0 | 3515.9 | 11394.3 |
| | Asset Incentive Ratio | 0.40 | 0.30 | 0.07 | 0.17 | 0.37 | 0.50 | 0.70 |
| | Equity Incentive Ratio | 0.29 | 0.24 | 0.03 | 0.09 | 0.23 | 0.42 | 0.60 |

## III. Incentives, Asset Risk-Taking, and the Financial Crisis of 2007/08

The Board of Governors of the Federal Reserve System (2011) begins its review of incentive compensation practices with the simple statement: "Risk-taking incentives provided by incentive compensation arrangements in the financial services industry were a contributing factor to the financial crisis that began in 2007" (p. 1). Even some of those whose pay is being heavily regulated seem to agree that compensation systems played a role.[17]

In this section, we study the relation between incentives of CEOs and asset risk-taking before the crisis of 2007/08.

### III.1. Hypotheses, Empirical Strategy, and Data

It is important to note at the outset that shareholders in principle (ex-ante) welcomed the asset risk-taking that later turned out to be harmful to the health of their financial institutions. For example, holding AAA tranches of securitized loans was appealing to shareholders for two reasons. First, these tranches paid extra yields over and above the typical AAA investments. Second, whether held on or off the balance sheet, these investments did not require backing by enhanced equity capital.[18] Thus, to the extent that there are factors apart from material rewards that make CEOs act in the interest of shareholders, part of the asset risk-taking in banks, as in other corporations, will not be explained by direct monetary incentives of CEOs. Functioning alignment of CEO actions with shareholder interests in financial institutions thus generates a baseline amount of asset risk-taking.

What we are interested in is whether part of the variation in asset risk-taking beyond this baseline level can be explained by incentives embedded in the equity and stock option holdings of managers. Building on the earlier considerations (see Section I.E), we test three main hypotheses regarding

[17] For example, in May 2008, PricewaterhouseCoopers and the Economist Intelligence Unit conducted a global survey of financial services industry executives and commentators. Asked which factors have created the conditions for the credit/banking crisis, only 31% of survey participants put the blame on "monetary policy," but an impressive 70% on "reward systems." See PricewaterhouseCoopers (2008).

[18] Whether equity is "expensive" is debated hotly. These discussions notwithstanding, it is a fact that most practitioners did believe that holding more equity was not desirable.

this relation.

***Hypothesis 1:*** Asset Volatility Vega is positively associated with asset risk-taking.

***Hypothesis 2:*** Asset Delta is negatively associated with asset risk-taking.

***Hypothesis 3:*** The Asset Incentive Ratio is positively associated with asset risk-taking.

Our empirical strategy to test these hypotheses is straightforward: We run cross-sectional regressions with a measure of asset risk-taking as the dependent variable. Risk-taking incentives, governance features, and other firm-specific variables serve as explanatory variables. We identify sub-industries with dummies in our regressions.

We first explain the choice of our dependent variable (Subsection III.1.1). We next describe the explanatory variables (Subsection III.1.2). Subsection III.1.3 then discusses issues related to endogeneity problems in the relationship between asset risk-taking incentives and asset risk-taking.

An overview of all dependent and explanatory variables is contained in Table III.

### III.1.1. Dependent Variable: Write-downs

Like other studies (Coles, Daniel, and Naveen, 2006; Guay, 1999; Hayes, Lemmon, and Qiu, 2011; Sanders and Hambrick, 2007), our empirical tests rely upon ex-post evidence of asset risk-taking. The idea we appeal to is that for a given expected project value, a CEO with higher incentives to take risk will be willing to tolerate a greater spread in potential outcomes. Because the financial crisis exposed the downside of the investments that banks undertook in prior years, the *write-downs* form an indication of the degree of asset risk-taking.[19] Because we aim to capture as broadly as possible the potential downsides, we focus on write-downs during the period 2007Q3-2008Q4. We

[19] Write-downs, of course, also cover simply bad business choices, even those that were not considered risky ex-ante. For example, the practice of making "Ninja" (no income, no job, no asset) loans on the sheer hope that real estate prices would continue climbing was arguably ex-ante questionable. But not all risks that were taken can be labeled as ex-ante bad. Related to this question, there is some discussion as to just how much CEOs suffered from the crisis. On the one hand, Bebchuk, Cohen, and Spamann (2010) show that management teams in the case of Bear Stearns and Lehman Brothers were able to cash out large amounts of bonus compensation before the crisis. On the other hand, some evidence suggests that in general CEOs did not take actions they thought would be on average value-destroying and that they did not, on average, anticipate the crisis. For example, they did not sell their own shares prior to the crisis, see Fahlenbrach and Stulz (2011).

**Table III.** Variable description.

| | |
|---|---|
| *Write-downs* | |
| Write-downs (USD mill.) | The losses incurred by the financial institutions during the financial crisis period 2007Q3-2008Q4. |
| Write-downs scaled by total assets | Write-downs divided by a company's total assets. |
| *CEO compensation* | |
| Equity Volatility Vega (1000 USD) | The dollar change in the CEO's wealth for a 0.01 change in the standard deviation of returns. |
| Equity Delta (1000 USD) | The dollar change in the CEO's wealth with respect to a one percent change in the underlying stock price. |
| Equity Incentive Ratio | Equity Volatility Vega divided by Equity Delta |
| Asset Volatility Vega (1000 USD) | The dollar change in the CEO's wealth for a 0.01 change in the standard deviation of firm value returns. |
| Asset Delta (1000 USD) | The dollar change in the CEO's wealth with respect to a one percent change in the underlying firm value. |
| Asset Incentive Ratio | Asset Volatility Vega divided by Asset Delta |
| *Firm Characteristics* | |
| Market Cap. (USD mill.) | The market capitalization of the company |
| Total assets (USD mill.) | Total assets on the company's balance sheet. |
| Book-to-Market ratio (%) | Book value of assets over market value of assets. |
| Book leverage (%) | Book leverage = 1 - (book value of equity / book value of assets) |
| Market leverage (%) | Market leverage = 1 - (market value of equity / market value of the financial institution) where market value of equity equals the number of shares times end-of-year stock price and the market value of the financial institution equals the market value of equity plus the book value of liabilities |
| *Governance* | |
| Percentage independent directors (%) | The fraction of directors on a board classified as independent. |
| Tenure | The number of years the CEO has been in office. |
| Governance index | The number of anti-takeover provisions a company has in place. |

collect write-downs data for all U.S. financial institutions for which they are available and for which we have compensation data from ExecuComp.

**Description of Write-Downs:** For the largest U.S. financial institutions these write-downs are available from Bloomberg, covering write-downs, losses, and loan-loss provisions. For the smaller U.S. financial institutions for which Bloomberg does not record write-downs we consult the companies' proxy filings (10-K and 10-Q). In particular, we identify the following components from the SEC filings in order to be as consistent as possible with the figures reported by Bloomberg: (1) Write-downs which are explicitly referred to as such. They cover charge-offs on loans (conditional on the fact that these are not included in the loan loss provisions). Furthermore, as a consequence of the financial crisis, some companies had to abandon certain development projects which led to rising severance charges. These are typically reported as specific write-downs related to the crisis. (2) Loan loss provisions are charges or expenses against income and loans which are deemed to be uncollectible due to the impact of the credit deterioration during the crisis period. (3) Subprime losses appear when companies directly state that certain losses have occured specifically due to, e.g., investments in the subprime mortgage backed security market or due to the bankruptcies of Fannie Mae and Freddie Mac. (4) Impairment charges (or impairment on securities) are non-temporary impairments on held to maturity and available for sale securities. This is sometimes referred to as losses on trading securities or impairment on real estate investments. (5) Credit losses which are directly referred to as such but are not included in the loan loss provisions.

We use both the logarithm of write-downs to study the absolute (dollar level) of asset risk-taking (controlling, of course, for firm size in various ways) and write-downs scaled by total assets to investigate the relative level of asset risk-taking.

**Advantages and Disadvantages of Write-downs and Alternative Measures**: Write-downs form a natural proxy for the ex-ante asset risk-taking of financial institutions precisely because they are not only realized losses, but also unrealized losses. Even if assets that are held to maturity in the end do not lead to an actual loss, the fact that banks had to take write-downs on them indicates their ex-ante riskiness. Also, while concrete policy choices can in principle be read from banks' financial reports, the discretion banks have in classifying certain assets puts bounds on the exactness of information obtained from these data.[20] Write-downs are a summary variable that captures all these facets in a relatively straightforward way.

Nonetheless, write-downs bring with them some limitations which need to be borne in mind. First, write-downs are accounting data. They are not always completely clearly and unambiguously described in company reports.[21]

Second, and related, firms have discretion of when to announce which write-downs. Also, in October 2008, the SEC allowed banks to switch from mark-to-market accounting to hold-to-maturity accounting. We cover a relatively wide data period, but it is still possible that some write-downs that were announced were not "fair value" losses, or that some losses have not yet been recorded as write-downs.[22] Some studies report that banks use accounting discretion to understate the impairment of their real estate related assets (Huizinga and Laeven, 2012). Others find the opposite, namely, that poorly-performing banks overstate unrealized losses ("take a bath") in order to show higher earnings the following year (Fiechter and Meyer, 2009). Yvas (2011) documents that higher corporate governance quality was associated with timelier write-downs in the time period 2007/08. His study takes it as given that the cumulated actual write-downs and those implied by benchmarking devaluations to credit indices are identical per the end of 2008. What is certain is that investors demonstrated a particularly keen interest in write-downs during the financial crisis, which, together

[20] Moreover, investing in one asset class, for example, MBS securities, may have meant simultaneously shifting out of another class; moreover, MBS investments often were acquired by banks in the process of securitization, so that the choice of banks was, in fact, multi-dimensional. Understanding these choices would probably require jointly modeling all policy choices.

[21] Naturally, one can debate in each and every case which parts of the announced overall write-downs should be included in the analysis, and it can be difficult to precisely disentangle some of the above categories from each other. We use the sum of all losses associated to the crisis. For a discussion of the challenges and opportunities of accounting in the financial crisis, see Ryan (2008).

[22] For example, write-downs may be overestimated because some firms may have been pushed by government authorities to "come clean." Conversely, write-downs may be underestimated due to the fact that some financial institutions were bailed out just for the bailout funds to flow through indirectly to other banks which could otherwise have ended up in deep trouble.

with accounting standards (in particular, FAS 157 - Fair Value Measurement which became effective for annual periods beginning on or after November 15, 2007) requiring detailed disclosure, is likely to have reduced opportunities for manipulation.[23]

### III.1.2. Explanatory Variables

**Incentives.** We use the incentive measures developed above.

**Company Characteristics.** We obtain company-specific information from the Compustat Fundamental and Bank Annual databases. As control variables in the main analysis we include proxies for firm size (the logarithm of market capitalization), the ratio of the book value of assets over the market value of assets (as a proxy for the companies potential investment opportunities), and market or book leverage.

**Corporate Governance Features.** We also control for a set of corporate governance variables including (a) *CEO Tenure* which measures the number of years the CEO has been in office and (b) *Percentage of independent directors* which is the fraction of directors on a board classified as independent. These data are from ExecuComp and Riskmetrics, supplemented by hand-collected data where possible. (c) The *Governance index* is the number of anti-takeover provisions a company has in place (Gompers, Ishii, and Metrick, 2003). A higher value of the Governance index is regarded as less shareholder-friendly governance.

[23] It is possible that despite diligent reading of the proxy statements available, we missed some write-downs in our data collection. If all institutions report truthfully, the losses must ultimately show up in net income. The drawback of using net income over assets is that it is a more noisy measure of ex-ante asset risk-taking, as it also comprises a lot of other activities of the financial institutions, including gains and losses from other business lines unrelated to the crisis. Nonetheless, the overall results with net income over assets as the dependent variable are similar to those for write-downs.

### III.1.3. Endogeneity Concerns

The cross-sectional regressions we employ do not allow us to strictly identify a causal effect. Our results are correlations, and we interpret them as such. Various features of our analysis may ameliorate endogeneity concerns, however.

*First*, as for reverse causation, we are considering the relationship between incentives (and other variables) in the years 2003-2006 and outcomes over the years 2007-2008. We do not have information on the decision criteria boards used to allocate incentive packages in a given year. It is conceivable that incentive packages of a given year include options given as a reward for undertaking risky deals in earlier years; this would imply an upward bias in the respective estimates. However, this concern is much less likely to apply to incentives in 2003. Incentive packages relevant for that year may, of course, include stock and options given as a reward for other asset risk-taking in prior years, but these potential earlier risky activities do not include the investments that led to write-downs in the financial crisis. Using these earlier years is also attractive because the vast majority of deals related to the subprime and mortgage backed security market originated in the early part of the decade, not in 2006. This is illustrated in Figure 3. While subprime mortgages have been used for a long time, the "take-off" of the market occurred around 2002/3 (e.g., Hässig (2009) for the case of UBS).

*Second*, asset substitution incentives of managers explicitly take into account leverage. Our regressions, therefore, show the relation between incentives and asset risk-taking for a given level of leverage. The ideal situation would be to jointly model the setting of incentives and the choices of the CEO to take asset risk through increasing the firm value return volatility and financial risk through higher firm leverage (in the spirit of Coles, Daniel, and Naveen (2006)).[24] Unfortunately, we cannot do this in our cross-sectional setup. Market leverage may, however, be subject to fewer concerns because it fluctuates passively simply because of changes in stock price performance (Welch,

[24]Indeed, not only the level of debt but also the maturity is endogenous Brockman, Martin, and Unlu (2010). As explained earlier, in line with Guay (1999) we use identical debt maturities for all financial institutions.

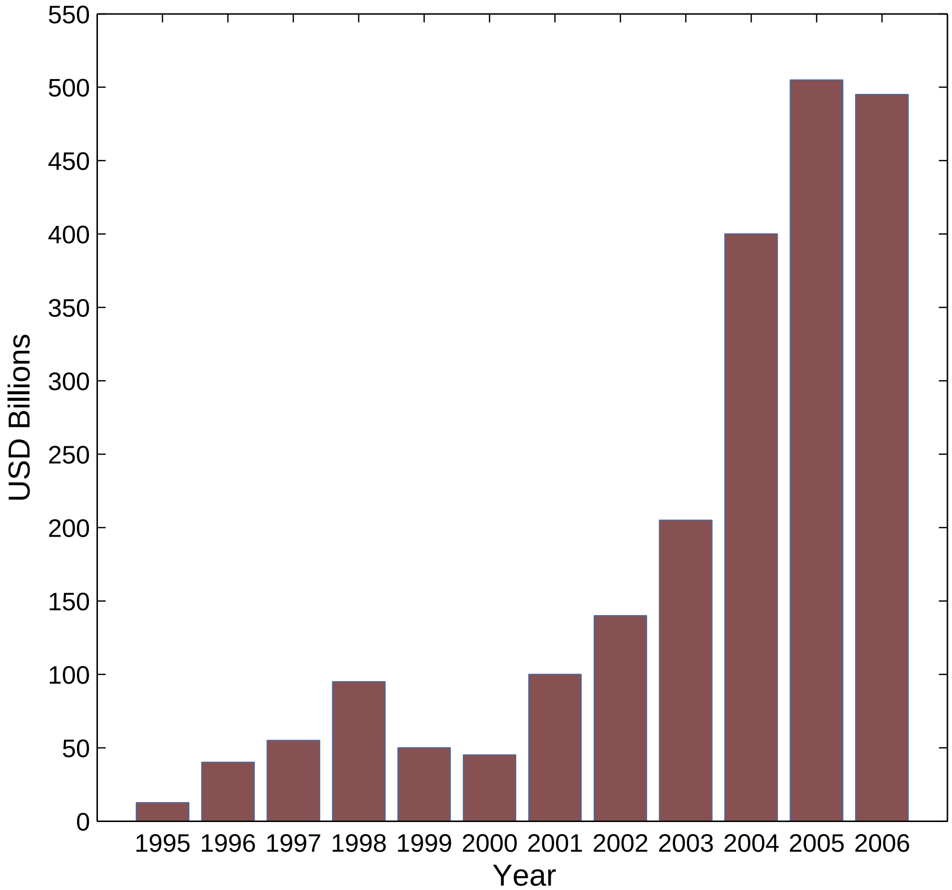


**Figure 3.** The development in subprime mortgage securitization over the years 1995-2006. Source: Inside Mortgage Finance, http://www.insidemortgagefinance.com.

2004). Thus, the endogeneity of leverage may be less of a concern given that we use market leverage throughout. All our results hold more strongly with book leverage.

*Third*, we control for important firm-level variables as well as industry dummies. We also control for several governance features that may be correlated both with incentives and risk-taking and are likely to capture important differences between firms. Of course, this still does not rule out the possibility that a positive association between risk-taking incentives and write-downs could arise because of omitted factors that are positively correlated with both variables.[25] It is equally possible that we are underestimating the relationship between risk-taking incentives and poor outcomes.[26] Despite several controls, our analysis cannot definitively rule out these concerns. (We will briefly comment on instrumental variables regressions that yield results consistent with a causal effect of incentives on asset risk-taking, but our sample is simply too small to reliably employ this identification strategy.)

## III.2. Regression Results

### III.2.1. Main Findings

During the credit crisis period, the companies in our sample had on average write-downs of around US$6.6bn, which implies write-downs of around 5.5% of their total assets (averaging assets over the years). See Table I; monetary values are in 2008 dollars. The heterogeneity in asset risk-taking also shows in the standard deviation of write-downs and scaled write-downs.

Table IV presents the main regression results. Panel A relates log write-downs to the Asset Incentive Ratio and the Equity Incentive Ratio, respectively. We observe that the Asset Incentive Ratio in 2003, 2004, and 2005 is positively and significantly (in one case only borderline so) related to write-downs. In 2006, we find no significant relationship. Given that the vast majority of subprime deals was done before 2006, it is perhaps not too surprising that we do not find a robustly significant relationship between incentives in 2006 and write-downs; however, when controlling for governance measures, this relationship does become significant (the results are not shown to conserve space).

[25] For instance, board competence is unobserved. Some may argue that less competent boards are more easily captured by the CEO and may, therefore, grant an excessive number of options to CEOs. Moreover, less competent boards are less able to monitor investments and may provide worse advice to the CEO. These two factors may combine into a cross-sectional positive relationship between risk-taking incentives and write-downs. Or, the least talented CEOs (who choose the worst projects on average) may be inclined to self-select into the firms with the highest risk-taking incentives, to occasionally "hit the jackpot."

[26] For example, if a company has a culture of risk-taking, it may attract risk-seeking individuals and may, thus, need to provide incentives with lower Vega than other firms. At the same time, these companies may, indeed, engage in a lot of risk-taking simply because the manager likes risk, which, in the case of the subprime crisis resulted in large write-downs.

By contrast, we find no significant association between write-downs and the Equity Incentive Ratio in any year. Indeed, for this measure the point estimates for 2003 and 2006 are of equal magnitude, but with opposite signs.

A similar picture emerges in Panel B. Here, we consider write-downs divided by total assets as a measure of relative asset risk-taking. In 2003, 2004, and 2005, the Asset Incentive Ratio is positively and (in one case borderline) significantly related to write-downs scaled by total assets. As before, for 2006 we do not obtain significant results in these regressions. By contrast, when we relate write-downs scaled by total assets to Equity Volatility Vega, we find some (borderline) significance in on year, 2003, but none at all in the other years. One difference between the results for relative asset risk-taking (write-downs scaled by firm size) and absolute asset risk-taking (log write-downs) is that the regressions for the latter generally show greater explanatory power than for the former, in terms of R-squared. Indeed, only asset incentives offer explanatory power for write-downs scaled by assets; by contrast, there is no size effect in relative asset risk-taking, for example.

In Table V we expand the regression specification in two ways. Columns (1)-(3) and (5)-(7) of Table V add governance variables. Including these variables is likely to ameliorate potential endogeneity concerns. The Asset Incentive Ratio remains consistently significant in all regressions. To conserve space we only show the results for 2003, but the results for the other years are similar and, with governance in the regressions, the results are also more significant for 2006. The Equity Incentive Ratio remains insignificant throughout when controlling for governance. (As another way to tackle the potential endogeneity concerns, we have also experimented with instrumental variables. These results are reported below.)

Finally, columns (4) and (8) show the Asset Volatility Vega and the Asset Delta separately in the regression instead of the Asset Incentive Ratio. We observe that Asset Volatility Vega is positively related to log write downs, while Asset Delta is negatively and significantly associated to write-downs consistent with Hypothesis 1 and 2. Moreover, these results are also similar to what Knopf, Nam, and Thornton (2002) find for corporate hedging activities. As such, our findings confirm that

**Table IV.** Asset risk-taking and managerial incentives: Main results

Regressions of log write-downs and write-downs scaled by total assets on CEO incentives and firm-level variables. Write-downs are for 2007Q3-2008Q4. The explanatory variables are those from the years stated in the columns. The variables are described in Table III. All variables are winsorized at the 2nd and 98th percentile. The Asset Incentive Ratio and the Equity Incentive Ratio represent Asset Volatility Vega divided by Asset Delta and Equity Volatility Vega divided by Equity Delta, respectively. Robust $t$-statistics are given in the parentheses and ***, ** and *, denote the statistical significance of the parameter estimates at the 1%, 5% and 10% levels, respectively.

Panel A: *Log write-downs*

| | (2003) | (2004) | (2005) | (2006) | (2003) | (2004) | (2005) | (2006) |
|---|---|---|---|---|---|---|---|---|
| Asset Incentive Ratio | 1.11** | 1.01 | 1.33** | 0.83 | | | | |
| | (2.14) | (1.53) | (2.06) | (1.23) | | | | |
| Equity Incentive Ratio | | | | | 0.57 | -0.09 | 0.38 | -0.49 |
| | | | | | (1.11) | (-0.21) | (0.95) | (-1.05) |
| Log market cap | 1.06*** | 1.07*** | 1.13*** | 1.20*** | 1.08*** | 1.13*** | 1.10*** | 1.20*** |
| | (10.37) | (10.65) | (12.65) | (15.90) | (10.32) | (11.18) | (11.27) | (15.09) |
| Book-to-market | 0.82 | 1.74 | 0.40 | 1.64* | 1.66* | 2.60** | 1.43* | 2.29*** |
| | (0.81) | (1.40) | (0.40) | (1.70) | (1.77) | (2.45) | (1.70) | (2.92) |
| Market leverage | 5.87*** | 4.91*** | 4.93*** | 3.71*** | 6.90*** | 5.57*** | 6.14*** | 4.13*** |
| | (4.09) | (3.66) | (3.96) | (3.78) | (4.02) | (3.28) | (3.23) | (4.44) |
| Constant | -8.06*** | -7.87*** | -7.95*** | -8.09*** | -9.01*** | -8.70*** | -8.77*** | -8.24*** |
| | (-6.19) | (-6.76) | (-7.95) | (-9.86) | (-6.25) | (-6.73) | (-6.45) | (-10.04) |
| Industry fixed effects | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| Observations | 95 | 99 | 99 | 119 | 95 | 99 | 99 | 119 |
| Adj. $R^2$ | 0.73 | 0.75 | 0.75 | 0.75 | 0.72 | 0.74 | 0.74 | 0.75 |

Panel B: *Write-downs scaled by total assets*

| | (2003) | (2004) | (2005) | (2006) | (2003) | (2004) | (2005) | (2006) |
|---|---|---|---|---|---|---|---|---|
| Asset Incentive Ratio | 0.09*** | 0.10** | 0.07 | 0.03 | | | | |
| | (3.71) | (2.27) | (1.50) | (1.11) | | | | |
| Equity Incentive Ratio | | | | | 0.06 | 0.00 | 0.01 | -0.01 |
| | | | | | (1.47) | (0.18) | (0.38) | (-0.39) |
| Log market cap | -0.01 | -0.01 | -0.01 | -0.00 | -0.01 | -0.01 | -0.01 | -0.00 |
| | (-1.47) | (-1.29) | (-0.66) | (-0.20) | (-1.27) | (-0.90) | (-0.71) | (-0.28) |
| Book-to-market | -0.05 | 0.02 | -0.02 | 0.02 | 0.02 | 0.10 | 0.04 | 0.05* |
| | (-1.11) | (0.35) | (-0.52) | (0.81) | (0.37) | (1.62) | (0.99) | (1.83) |
| Market leverage | 0.12 | 0.02 | 0.03 | 0.02 | 0.20 | 0.09 | 0.08 | 0.03 |
| | (0.92) | (0.28) | (0.29) | (0.44) | (1.54) | (0.84) | (0.77) | (0.67) |
| Constant | 0.06 | 0.09 | 0.05 | 0.01 | -0.01 | 0.01 | 0.02 | 0.01 |
| | (0.90) | (1.64) | (1.00) | (0.39) | (-0.17) | (0.14) | (0.33) | (0.22) |
| Industry fixed effects | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| Observations | 95 | 99 | 99 | 119 | 95 | 99 | 99 | 119 |
| Adj. $R^2$ | 0.12 | 0.18 | 0.10 | 0.06 | 0.10 | 0.11 | 0.06 | 0.05 |

incentives were related to risk-relevant activities in the time ahead of the financial crisis just like in other times.

Overall, the findings broadly provide support for Hypotheses 1, 2 and 3, confirming the idea of a link between asset risk-taking and incentives to take asset risk. By contrast, we do not find a significant relationship between equity incentives and asset risk-taking. Of course, given that the Asset Incentive Ratio and the Equity Incentive Ratio are correlated, we do find some relationship between the Equity Incentive Ratio and asset risk-taking, too, but the relation is overall much weaker. Thus, our analysis documents that, in order to explain asset risk-taking, it can be important to use asset incentives instead of equity incentives.[27]

### III.2.2. Additional Findings

As can be seen in Table V, we find no robust association of any of the governance characteristics with write-downs, controlling for incentives to take asset risk. That we detect no significant relationship between director independence and write-downs may be the result of several countervailing factors. On the one hand, boards acting more strongly on behalf of shareholders may have pushed CEOs to engage in more asset risk-taking (thus implying a positive relationship).[28] On the other hand, such boards may also have been more prudent in avoiding the worst investments. Moreover, board independence does not directly capture board competence (Fernandes and Fich, 2013).

Some additional results are not tabulated to conserve space. First, we considered two instruments for incentives to try to get closer to a causal estimate of the impact of incentives on asset risk-taking:

[27] From society's point of view, the relationship between write-downs and compensation structures was accentuated because the downsides of financial institutions' investments resulted in significant external costs presumably not taken into account by shareholders. Thus, a natural future research question is whether and how incentive systems at financial institutions may encourage managers of these companies to take into account (to a greater extent than is arguably common today) the external effects of their actions. Some work exists that has begun addressing related issues. For example, on optimal incentive design in the presence of government guarantees see John, Saunders, and Senbet (2000), John, Mehran, and Qian (2010), Bebchuk and Spamann (2010), and Bolton, Mehran, and Shapiro (2011).

[28] It is also possible that more independent boards forced executives to disclose write-downs earlier or higher. This argument only holds if one posits that despite the strict accounting regime and the eagerness of investors to monitor developments at financial institutions, some banks were able to manipulate the total amount of write-downs in the six quarters considered here.

**Table V.** Asset risk-taking and managerial incentives: Additional results

Regressions of log write-downs and write-downs scaled by total assets on CEO incentives, firm-level and governance variables. Write-downs are for 2007Q3-2008Q4. The explanatory variables are from 2003. The variables are described in Table III. All variables are winsorized at the 2nd and 98th percentile. Monetary values are measured in 2008 dollars. Robust $t$-statistics are given in the parentheses and ***, ** and *, denote the statistical significance of the parameter estimates at the 1%, 5% and 10% levels, respectively.

| | *Log write-downs* | | | | *Write-downs scaled by total assets* | | | |
|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| Asset Incentive Ratio | 1.08** | 1.34** | 1.04* | | 0.09*** | 0.10*** | 0.10*** | |
| | (2.21) | (2.53) | (1.82) | | (3.07) | (3.22) | (3.08) | |
| Log Asset Volatility Vega | | | | 0.56*** | | | | 0.02 |
| | | | | (3.78) | | | | (1.65) |
| Log Asset Delta | | | | -0.61*** | | | | -0.03** |
| | | | | (-3.13) | | | | (-2.18) |
| Log market cap | 1.09*** | 1.08*** | 1.13*** | 1.10*** | -0.02 | -0.02 | -0.01 | -0.00 |
| | (9.44) | (9.67) | (9.65) | (10.69) | (-1.29) | (-1.25) | (-1.01) | (-0.57) |
| Book-to-market | 1.49 | 1.70* | 2.33** | 0.58 | -0.02 | -0.02 | -0.02 | -0.01 |
| | (1.54) | (1.75) | (2.30) | (0.58) | (-0.40) | (-0.43) | (-0.27) | (-0.17) |
| Market leverage | 5.55*** | 5.51*** | 5.30*** | 6.06*** | 0.14 | 0.14 | 0.14 | 0.18 |
| | (3.82) | (3.26) | (2.94) | (3.99) | (0.95) | (0.93) | (0.87) | (1.27) |
| Perc. Ind. Directors | 0.00 | | | | 0.00 | | | |
| | (0.10) | | | | (1.07) | | | |
| Tenure | | 0.01 | | | | -0.00 | | |
| | | (0.30) | | | | (-0.15) | | |
| G-index | | | 0.07 | | | | -0.00 | |
| | | | (1.29) | | | | (-0.13) | |
| Constant | -8.34*** | -8.57*** | -9.53*** | -6.95*** | 0.00 | 0.05 | 0.03 | 0.08 |
| | (-5.02) | (-5.69) | (-5.86) | (-4.89) | (0.02) | (0.73) | (0.41) | (0.98) |
| Industry fixed effects | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| Observations | 74 | 78 | 75 | 95 | 74 | 78 | 75 | 95 |
| Adj. $R^2$ | 0.76 | 0.75 | 0.77 | 0.73 | 0.15 | 0.15 | 0.16 | 0.11 |

(1) past cumulative returns and (2) lagged stock return volatility. High previous-year cumulative returns are likely to induce the board to reduce risk incentives going forward (because they suggest that a lot of risk was taken previously), but should not be directly correlated with asset risk-taking in the years before the crisis.[29] Similarly, higher equity volatilities (measured over 5 years prior to the year when we measure the incentives) are correlated with current incentives because they indicate that it was relatively cheap to grant CEOs stock options, and the historical volatility can arguably be excluded from regressions explaining asset risk-taking just before the crisis.[30] With these instruments we obtain results supportive of a significant, positive causal impact of the Asset Incentive Ratio on write-downs.[31] However, the reliability of 2SLS may be limited with this small sample size, so we do not emphasize these results.

The regression results continue to hold when we consider book leverage. In the spirit of Edmans, Gabaix, and Landier (2009), we scaled Delta by total compensation. The results continue to hold. Finally, although this is not the focus of the paper, we consider performance of banks in the crisis, not asset risk-taking, as the dependent variable. We find that financial institutions with higher Asset Volatility Vegas had lower stock returns during the crisis period, whereas those with higher Asset Deltas had higher returns. Fahlenbrach and Stulz (2011) had found no relationship of stock returns with equity incentives. These results again confirm that asset incentives capture different features than the equity incentives.

[29] This instrument is often used in studies of risk-taking (Armstrong and Vashishtha, 2012; DeYoung, Peng, and Yan, 2012).

[30] Indeed, studies aiming to explain risk-taking do not generally use historical volatility as an explanatory variable because there is no reason to expect a direct effect.

[31] For example, for 2003 and log write-downs, the first-stage F-statistic is 13.34, above the critical value of 11.59 for two instruments suggested by Stock, Wright, and Yogo (2002), ameliorating weak instruments concerns; the Hansen-Sargan J statistic yields a p-value of 0.91, implying that, conditional on one instrument fulfilling the exclusion restriction, the other instrument is also likely to be valid. A Hausman test (in the two instruments version) has a p-value of just below 0.1, suggesting that the data do not actually reject the use of OLS in favor of 2SLS (the point estimates are not very different).

## IV. Concluding Remarks

This paper offers, to our knowledge, the first investigation of the quantitative importance of managerial incentives to take asset risk. It is motivated by the observation that, while the notion that asset risk-taking incentives depend on leverage has been a cornerstone of corporate finance since Jensen and Meckling (1976), the measures of equity incentives typically employed in research do not explicitly incorporate this idea. The asset incentives we calculate, instead, reflect the intuition that variation of leverage between companies needs to be taken into account when trying to understand managerial incentives to take risk. Each incentive measure should be used in the context where it is appropriate.

We show that the conceptual idea that asset incentives differ from equity incentives is also relevant quantitatively. We document three main results. *First*, incentives to take asset risk can be large compared to incentives to increase the value of assets; this provides a contrast to the fact that incentives to take equity risk are usually small compared to incentives to increase the stock price. *Second*, stock-holdings can also induce substantial asset risk-taking incentives; thus, the proposal, often heard in practice, to compensate CEOs mostly with stock rather than stock options, in order to rein in risk-taking incentives does not apply so cleanly anymore. *Third*, in our empirical application in the context of the financial crisis, asset incentives possess considerable explanatory power for asset risk-taking; using equity incentives, one would instead erroneously conclude that managerial incentives were unrelated to the asset risks that financial institutions took in the years before the financial crisis 2007/08.

These results may prove helpful for future studies on incentives and risk-taking and may aid boards in evaluating the incentives conveyed by equity-based compensation.

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# Appendix A: Asset Volatility Vega From Stock Options

Consider the compound option pricing model described in Section II.II.1. Our task is to calculate the sensitivity of the compound option pricing formula for a call option with respect to the underlying asset return volatility. Geske (1979) presents a formula for this derivative, which we denote by "Geske Vega,"

$$\begin{aligned} \text{Geske Vega} &= \frac{\partial CC(V, D, r, T_{CC}, T_D, \sigma_V, K)}{\partial \sigma_V^2} \cdot \frac{d\sigma_V^2}{d\sigma_V} \\ &= \frac{N_2(h + \sigma_V \sqrt{T_{CC}}, k + \sigma_V \sqrt{T_D}, \sqrt{T_{CC}/T_D})}{N_1(k + \sigma_V \sqrt{T_D})} D e^{-rT_D} \varphi(k) \sqrt{T_D}. \end{aligned} \tag{A.1}$$

However, we do not rely on this formula for computing the Asset Volatility Vega we use in our analysis. Our motivation not to employ this formula is that the Black-Scholes model is a special case of the compound option model when the firm's debt, $D$, goes to zero (this observation is also noted by Geske (1979)). Indeed, in the limit as $D \to 0$, the compound option price converges to the Black-Scholes price. Therefore, we would also expect that, for zero debt, the Asset Volatility Vega from the compound option model should collapse into the formula for the Asset Volatility Vega (and, thus, in this special case the Equity Volatility Vega) from the Black-Scholes model. However, the formula for the Geske Vega suggests, counter-intuitively, that in the limit when $D \to 0$ the Asset Volatility Vega given by the compound option pricing model would converge to zero.

In Proposition 1, we derive a different expression for the vega for a compound call option.

***Proof of Proposition 1.*** Taking the derivative of a call option in the compound option pricing model with respect to $\sigma_V$ gives us

$$\begin{aligned}
\frac{\partial CC}{\partial \sigma_V} &= V\left(\frac{\partial N_2(x, k+\sigma_V\sqrt{T_D}; \sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h+\sigma_V\sqrt{T_{CC}}} \cdot \frac{d(\sigma_V\sqrt{T_{CC}})}{d\sigma_V}\right.\\
&+ \left.\frac{\partial N_2(h+\sigma_V\sqrt{T_{CC}}, y; \sqrt{\frac{T_{CC}}{T_D}})}{\partial y}\Big|_{y=k+\sigma_V\sqrt{T_D}} \cdot \frac{d(k+\sigma_V\sqrt{T_D})}{d\sigma_V}\right)\\
&- De^{-rT_D}\frac{\partial N_2(h, y; \sqrt{\frac{T_{CC}}{T_D}})}{\partial y}\Big|_{y=k} \cdot \frac{dk}{d\sigma_V}\\
&+ \left[V\frac{\partial N_2(x, k+\sigma_V\sqrt{T_D}; \sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h+\sigma_V\sqrt{T_{CC}}}\right.\\
&- \left.De^{-rT_D}\frac{\partial N_2(x, k; \sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h} - Ke^{-rT_{CC}}\frac{dN(h)}{dh}\right]\frac{dh}{d\sigma_V}. \qquad \text{(A.2)}
\end{aligned}$$

In Lemma 1 (see below), we show that the expression in the square bracket of the last term in Equation (A.2) equals zero. This leaves us with

$$\begin{aligned}
\frac{\partial CC}{\partial \sigma_V} &= V\left(\frac{\partial N_2(x, k+\sigma_V\sqrt{T_D}; \sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h+\sigma_V\sqrt{T_{CC}}} \cdot \frac{d(\sigma_V\sqrt{T_{CC}})}{d\sigma_V}\right.\\
&+ \left.\frac{\partial N_2(h+\sigma_V\sqrt{T_{CC}}, y; \sqrt{\frac{T_{CC}}{T_D}})}{\partial y}\Big|_{y=k+\sigma_V\sqrt{T_D}} \cdot \frac{d(k+\sigma_V\sqrt{T_D})}{d\sigma_V}\right)\\
&- De^{-rT_D}\frac{\partial N_2(h, y; \sqrt{\frac{T_{CC}}{T_D}})}{\partial y}\Big|_{y=k} \cdot \frac{dk}{d\sigma_V}. \qquad \text{(A.3)}
\end{aligned}$$

Inserting the remaining derivatives used in Lemma 1 into Equation (A.3) we obtain the formula

$$\begin{aligned}\frac{\partial CC}{\partial \sigma_V} &= V\Bigg[\varphi(h+\sigma_V\sqrt{T_{CC}})N_1\left(\frac{k+\sigma_V\sqrt{T_D}-\sqrt{\frac{T_{CC}}{T_D}}(h+\sigma_V\sqrt{T_{CC}})}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right)\sqrt{T_{CC}} \\ &+ \varphi(k+\sigma_V\sqrt{T_D})N_1\left(\frac{h+\sigma_V\sqrt{T_{CC}}-\sqrt{\frac{T_{CC}}{T_D}}(k+\sigma_V\sqrt{T_D})}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right)\frac{d(k+\sigma_V\sqrt{T_D})}{d\sigma_V}\Bigg] \\ &- De^{-rT_D}\varphi(k)N_1\left(\frac{h-\sqrt{\frac{T_{CC}}{T_D}}k}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right)\frac{dk}{d\sigma_V},\end{aligned} \tag{A.4}$$

where

$$\frac{dk}{d\sigma_V} = -\frac{k}{\sigma_V} - \sqrt{T_D} \quad \text{and} \quad \frac{d(k+\sigma_V\sqrt{T_D})}{d\sigma_V} = -\frac{k}{\sigma_V}. \tag{A.5}$$

Now simplifying terms and using that $V\varphi(k+\sigma_V\sqrt{T_D}) = De^{-rT_D}\varphi(k)$ we obtain Proposition 1. ■

***Lemma 1.*** *Given the model assumptions,*

$$Q := V\frac{\partial N_2(x, k+\sigma_V\sqrt{T_D}; \sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h+\sigma_V\sqrt{T_{CC}}} - De^{-rT_D}\frac{\partial N_2(x, k; \sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h} - Ke^{-rT_{CC}}\frac{dN(h)}{dh} = 0 \tag{A.6}$$

***Proof of Lemma 1.*** By relying on the relation

$$N_2(h, k; \rho) = \int_{-\infty}^{h}\varphi(x)N_1\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)dx \tag{A.7}$$

as given in Geske (1979) (pages 79-80) and from the symmetry between $k$ and $h$, we have the following relations

$$\frac{\partial N_2(x,k+\sigma_V\sqrt{T_D};\sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h+\sigma_V\sqrt{T_{CC}}} = \varphi(h+\sigma_V\sqrt{T_{CC}})N_1\left(\frac{k+\sigma_V\sqrt{T_D}-\sqrt{\frac{T_{CC}}{T_D}}(h+\sigma_V\sqrt{T_{CC}})}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right) \tag{A.8}$$

and

$$\frac{\partial N_2(h+\sigma_V\sqrt{T_{CC}},y;\sqrt{\frac{T_{CC}}{T_D}})}{\partial y}\Big|_{y=k+\sigma_V\sqrt{T_D}} = \varphi(k+\sigma_V\sqrt{T_D})N_1\left(\frac{h+\sigma_V\sqrt{T_{CC}}-\sqrt{\frac{T_{CC}}{T_D}}(k+\sigma_V\sqrt{T_D})}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right) \tag{A.9}$$

and

$$\frac{\partial N_2(x,k;\sqrt{\frac{T_{CC}}{T_D}})}{\partial x}\Big|_{x=h} = \varphi(h)N_1\left(\frac{k-\sqrt{\frac{T_{CC}}{T_D}}h}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right) \tag{A.10}$$

and

$$\frac{\partial N_2(h,y;\sqrt{\frac{T_{CC}}{T_D}})}{\partial y}\Big|_{y=k} = \varphi(k)N_1\left(\frac{h-\sqrt{\frac{T_{CC}}{T_D}}k}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right), \tag{A.11}$$

where $\varphi$ denotes the standard normal probability density. Moreover, we use that $V\varphi(h+\sigma_V\sqrt{T_{CC}}) = \bar{V}e^{-rT_{CC}}\varphi(h)$ where $\bar{V}$ is the solution to Equation (5). This gives us

$$\begin{aligned} Q = \bar{V}e^{-rT_{CC}}\varphi(h)N_1\left(\frac{k+\sigma_V\sqrt{T_D}-\sqrt{\frac{T_{CC}}{T_D}}(h+\sigma_V\sqrt{T_{CC}})}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right) &- De^{-rT_D}\varphi(h)N_1\left(\frac{k-\sqrt{\frac{T_{CC}}{T_D}}h}{\sqrt{1-\frac{T_{CC}}{T_D}}}\right) \\ &- Ke^{-rT_{CC}}\varphi(h). \end{aligned} \tag{A.12}$$

Simplifying terms inside the standard normal distribution functions, we can write Equation (A.12) as follows

$$Q = e^{-rT_{CC}}\varphi(h)\left[\bar{V}N_1(\bar{k}(\bar{V}) + \sigma_V\sqrt{T_D - T_{CC}}) - De^{-r(T_D - T_{CC})}N_1(\bar{k}(\bar{V})) - K\right]. \tag{A.13}$$

By again relying on Equation (5) and on the definition of $\bar{V}$, we know that the term in square brackets is zero. Thus, our result is obtained. ■

***Remarks to the proof of Proposition 1.***

In some cases, researchers may be interested in analyzing how the firm value where the option is precisely at the money, $\bar{V}$, varies as the asset volatility varies. That is, we would wish to compute the term $\frac{d\overline{V}}{d\sigma_V}$. Although the term $\frac{dh}{d\sigma_V}$, in which $\bar{V}$ features, plays no role in the proof of Proposition 1 (the terms in front of it cancel out), we present the derivations for computing this derivative here for completeness.

Define the following function

$$f(x,y) = xN_1(k(x,y) + y\sqrt{T_D - T_{CC}}) - De^{-r(T_D - T_{CC})}N_1(k(x,y)) - K, \tag{A.14}$$

where

$$\tilde{k}(x,y) = \frac{\log(\frac{x}{D}) + (r - \frac{y^2}{2})(T_D - T_{CC})}{y\sqrt{T_D - T_{CC}}}. \tag{A.15}$$

Now, by relying on Equation (A.14) above we have that

$$f(\overline{V}, \sigma_V) = 0. \tag{A.16}$$

Since the function $f$ depends on $\sigma_V$ both through the direct effect but also indirectly through its effect on $\bar{V}$, we can consider the function $f$ as a function in these two variables. Writing up the dynamics using the chain-rule, we have

$$df(\overline{V}, \sigma_V) = \frac{\partial f}{\partial x}(\overline{V}, \sigma_V)d\overline{V} + \frac{\partial f}{\partial y}(\overline{V}, \sigma_V)d\sigma_V = 0, \tag{A.17}$$

which equivalently can be written as

$$\frac{d\overline{V}}{d\sigma_V} = -\frac{\frac{\partial f}{\partial y}(\overline{V}, \sigma_V)}{\frac{\partial f}{\partial x}(\overline{V}, \sigma_V)}. \tag{A.18}$$

We obtain

$$\frac{d\overline{V}}{d\sigma_V} = -\frac{De^{-r(T_D - T_{CC})}\varphi(\overline{k})\sqrt{T_D - T_{CC}}}{N_1(\overline{k} + \sigma_V\sqrt{T_D - T_{CC}})}, \tag{A.19}$$

where

$$\overline{k} = \tilde{k}(\overline{V}, \sigma_V). \tag{A.20}$$

Now it is straightforward to compute the term $\frac{dh}{d\sigma_V}$ in Equation (A.2). From,

$$h = \frac{\log(\frac{V}{\overline{V}}) + (r - \frac{\sigma_V^2}{2})T_{CC}}{\sigma_V\sqrt{T_{CC}}} \tag{A.21}$$

we obtain

$$\frac{dh}{d\sigma_V} = \frac{\frac{1}{\overline{V}\sqrt{T_{CC}}}\frac{De^{-r(T_D - T_{CC})}\varphi(\overline{k})\sqrt{T_D - T_{CC}}}{N_1(\overline{k} + \sigma_V\sqrt{T_D - T_{CC}})} - h}{\sigma_V} - \sqrt{T_{CC}} \tag{A.22}$$

■

We rely on the formula in Proposition 1 to measure the CEO asset risk-taking incentives we apply in the analysis. Analogously to the Asset Volatility Vega from stocks we divide the expression in Proposition 1 by 100 so that it measures the sensitivity of the option price with respect to a 0.01 change in the underlying asset return volatility, as we did it with the Equity Volatility Vega.

To confirm the intuitive benchmark result mentioned above, consider the case when the debt, $D$, tends to zero. Then, since $\bar{V}$ tends to the strike $K$ and $k$ tends to $+\infty$ it is clear that

$$\lim_{D \to 0} \frac{\partial CC}{\partial \sigma_V} = V\varphi(h + \sigma_V\sqrt{T_{CC}})\sqrt{T_{CC}} \tag{A.23}$$

which is precisely the vega of a call option in the Black-Scholes model and where $h$ is given in Equation (A.21).

Figure A-1 shows a comparison of the formulas for the *Asset Volatility Vega* given by the formula in Geske (1979) (see Equation (A.1)) with our Asset Volatility Vega (see Proposition 1) and the one computed by the first difference approximation for parameter values relevant for the analysis conducted in this paper and for varying levels of asset volatilities, $\sigma_V$ . We make two observations. First, as before, the analytical formula in given in Proposition 1 completely agrees with the first difference approximation. Second, there are substantial differences between the Asset Volatility Vega computed using the formula in Proposition 1 and the formula presented in Geske (1979) at common levels of the asset volatility. The difference (both in absolute and in relative terms) between the two approaches to compute the Asset Volatility Vega is particularly pronounced for moderate and low levels of asset volatility, which is commonly observed for financial institutions, though it remains substantial even at higher levels of asset volatility. That is, if we were to use the formula in Geske (1979), we would substantially underestimate the incentives to take asset risk as compared to our formula (and the difference approximation).

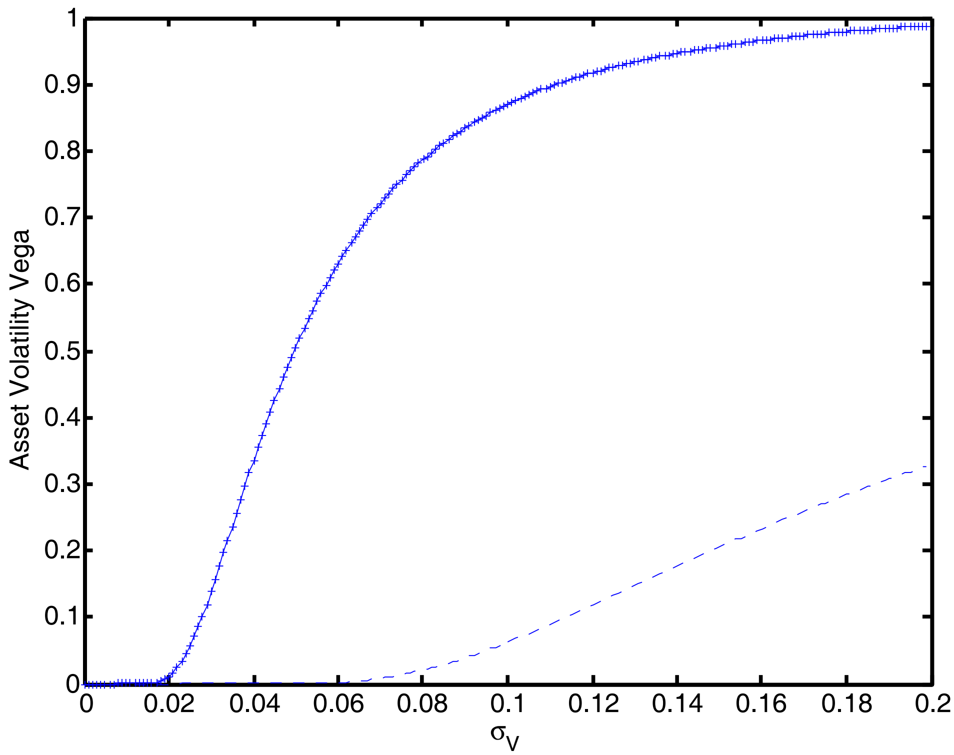


**Figure A-1.** Comparison of Asset volatility Vega to Geske's formula.

Comparison of Asset Volatility Vegas for varying asset volatility, $\sigma_V$, computed using three different approaches: The Geske approach (dotted line), our approach ("+") and the difference approximation (solid line). The parameters are $V = 100$, $K = 50$, $r = 0.04$, $T_D = 10$, $T_{CC} = 6$ and $D = 85$. $\Delta = 10^{-11}$ is used to calculate the difference approximation.

# Appendix B: Equity Delta and Equity Volatility Vega

Following Core and Guay (2002), we calculate Equity Delta and Equity Volatility Vega using the derivatives of the Black-Scholes formula (see Black and Scholes (1973)) with respect to the underlying stock and the stock return volatility respectively. We assume that the stock price, $S$, follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dW_t \tag{B.1}$$

under the historical measure. The parameters $\mu_S$ and $\sigma_S$ are assumed constant and we also assume that there exists a bank account which yields a constant interest rate $r$. Recall that the value of a plain vanilla call is given by

$$BS = Se^{-\delta T_D} N(d_1) - Ke^{-rT_D} N(d_2) \tag{B.2}$$

with

$$d_1(S, K, r, T_D, \sigma_S) = \frac{\ln(S/K) + ((r-\delta) + \sigma_S^2/2)T_D}{\sigma_S \sqrt{T_D}} \tag{B.3}$$

$$d_2(S, K, r, T_D, \sigma_S) = d_1(S, K, r, T_D, \sigma_S) - \sigma_S \sqrt{T_D}, \tag{B.4}$$

where $N$ denotes the cumulative distribution function of a standard normal random variable. $S$ denotes the stock, $K$ denotes the strike, $\sigma_S$ is the return volatility, $r$ represent the risk-free rate, $\delta$ denotes the dividend yield and $T_D$ represents the maturity of the option. $T$ corresponds to $T_{CC}$ in the Compound option pricing framework.

The *Equity Delta* of a single option can be computed according to

$$\text{Equity Delta} = \frac{\partial BS}{\partial S} \cdot (S/100) = e^{-\delta T_D} N(d_1) \cdot (S/100), \tag{B.5}$$

which measures the sensitivity of the option value with respect to a one percent change in the stock price.

The *Equity Volatility Vega* of a single option can be computed according to

$$\text{Equity Volatility Vega} = \frac{\partial BS}{\partial \sigma_S} \cdot (1/100) = e^{-\delta T_D} \varphi(d_1) S \sqrt{T_D} \cdot (1/100) \tag{B.6}$$

which measures the sensitivity of the option value with respect to a 0.01 change in the underlying stock return volatility and $\varphi$ denotes the density of a standard normal random variable.

As is standard in the literature, we assume that Equity Delta from stocks is equal to one (by construction) and that Equity Volatility Vega from stocks equals zero; see Guay (1999).

# Appendix C: Sample and Parameter Choices

Table C-1 shows the SIC codes we consider in the analysis. As described in the main text, we mostly follow Guay (1999) in our parameter choices. Table C-2 shows descriptive statistics for all the relevant variables.

To obtain an estimate of the firm value return volatility, $\sigma_V$, Guay (1999) proposes to rely on portfolio theory. Thus, the variance of the firm value can written as

$$\sigma_V^2 \quad = \quad X_{\text{debt}}^2 \sigma_{\text{debt}}^2 + X_{\text{equity}}^2 \sigma_{\text{equity}}^2 + 2 X_{\text{debt}} X_{\text{equity}} \text{Cov}(\sigma_{\text{debt}}, \sigma_{\text{equity}}), \tag{C.1}$$

where $X_{\text{debt}}$ and $X_{\text{equity}}$ are the weights of debt and equity in the firm's capital structure and $\sigma_{\text{equity}} \equiv \sigma_S$ is the annualized standard deviation of daily log stock returns. We use the same $\sigma_{\text{equity}}$ as we have used for computing Equity Volatility Vega and Equity Delta. For $\sigma_{\text{debt}}$, we use the annualized standard deviation of monthly (log) returns using the Merrill Lynch Bank of America corporate financial bond index using a five year period. Moreover, we follow Guay (1999) and set the correlation between equity and debt returns equal to one, $\text{Corr}(\sigma_{\text{debt}}, \sigma_{\text{equity}}) = \frac{\text{Cov}(\sigma_{\text{debt}}, \sigma_{\text{equity}})}{\sqrt{\sigma_{\text{debt}}^2 \sigma_{\text{equity}}^2}} = 1$, which implies that $\text{Cov}(\sigma_{\text{debt}}, \sigma_{\text{equity}}) = \sigma_{\text{debt}} \sigma_{\text{equity}}$ and, therefore,

$$\sigma_V^2 = X_{\text{debt}}^2 \sigma_{\text{debt}}^2 + X_{\text{equity}}^2 \sigma_{\text{equity}}^2 + 2 X_{\text{debt}} X_{\text{equity}} \sigma_{\text{debt}} \sigma_{\text{equity}}. \tag{C.2}$$

**Table C-1.** Industry classification.

SIC code 6211 includes some well-known investment banks and some brokers. For our main analysis, we keep these brokerage firms in the sample but exclude those brokers listed in the same SIC code as exchanges, SIC code 6200. While engagement in the subprime mortgage business may not have been at the core of the business strategy of those brokers in the SIC 6211 code, these companies nonetheless often did engage in such activities. Our regression results generally speaking hold also if we exclude all SIC 6211 firms, but this then also excludes firms such as Bear Stearns, Goldman Sachs, Merrill Lynch, and Morgan Stanley, clearly an undesirable sample restriction. We do not consider investment advisors (SIC code 6282). For other finance SIC codes that are not shown we do not have companies with compensation data.

| *Financial institutions* | *2-digit SIC* | *SIC Code* | *Financial Service Industry* |
|---|---|---|---|
| Depository Institutions | 60 | 6020 | Commercial Banks |
| | | 6035 | Federal Savings Institutions |
| | | 6036 | Savings Institutions, Except Federal |
| | | 6099 | Functions Related to Depository Banking |
| Nondepository Credit | 61 | 6111 | Federal Credit Agencies |
| Institutions | | 6141 | Personal Credit Institutions |
| | | 6159 | Miscellaneous Business Credit |
| | | 6162 | Mortgage Bankers and Loan Correspondents |
| | | 6172 | Finance Lessors |
| | | 6199 | Finance Services |
| Security Brokers and Dealers | 62 | 6211 | Security Brokers and Dealers |

**Table C-2.** Estimated parameters for the Compound option pricing model

Descriptive statistics of the parameters used in the computations of Asset Volatility Vega and Asset Delta. The summary statistics are average over the years 2003-2006. Per-share stock price denotes the average end-of-year stock price. Per-share book-value of debt denotes the book value of liabilities end-of-year. The risk-free interest rate is the yield on U.S. Treasury bonds with a maturity similar to maturity of the firms liabilities. Standard deviation of debt returns is the annualized standard deviation calculated on monthly (log) returns using the Merrill Lynch Bank of America corporate financial bond index using a five year period. Standard deviation of equity returns is calculated (from CRSP data) as the annualized standard deviation of daily log-returns over the past three years up to each year in our sample by assuming 250 trading days in the year. Est. std. dev. of returns on firm value denotes the estimated standard deviation of returns on the firm value. Weight of equity and Weight of debt are, respectively, the shares of equity and debt in the firm's capital structure. Implied per-share market value of assets is backed out using the Black-Scholes equation. Per-share market value of assets denotes sum of the per share end-of-year stock price and the per share book value of debt. Price-to-strike ratio is the implied per share firm value divided by the per share book value of debt. The variables are winsorized at the 2nd and 98th percentile on an annual basis. The term "q" denotes the quantile, i.e., 10q refers to the 10th percentile in the empirical distribution of the respective variable. All monetary values are expressed in 2008 dollars.

| Firm characteristics | Mean | Std. Dev. | 10q | 25q | 50q | 75q | 90q |
|---|---|---|---|---|---|---|---|
| Per-share stock price ($) | 42.0 | 22.7 | 17.9 | 25.6 | 37.3 | 52.8 | 69.6 |
| Per-share book-value of debt ($) | 238.3 | 254.3 | 61.2 | 103.9 | 160.4 | 282.4 | 409.7 |
| Risk-free interest rate (%) | 4.1 | 0.4 | 3.7 | 3.9 | 4.3 | 4.6 | 4.6 |
| Standard deviation of debt returns (%) | 3.1 | 0.1 | 2.9 | 2.9 | 3.0 | 3.2 | 3.2 |
| Standard deviation of equity returns (%) | 27.5 | 9.5 | 17.3 | 21.3 | 25.8 | 32.0 | 37.8 |
| Est. std. dev. of returns on firm value (%) | 5.7 | 2.8 | 4.0 | 4.5 | 5.1 | 5.9 | 7.1 |
| Weight of equity (%) | 10.4 | 6.9 | 5.6 | 7.4 | 9.0 | 10.5 | 14.3 |
| Weight of debt (%) | 89.4 | 7.9 | 85.6 | 89.4 | 91.0 | 92.6 | 94.4 |
| Imp. per-share market value of assets ($) | 215.6 | 200.4 | 67.1 | 108.1 | 156.9 | 256.0 | 373.3 |
| Per-share market value of assets ($) | 280.0 | 270.4 | 85.2 | 134.2 | 201.4 | 333.4 | 476.3 |
| Price-to-strike ratio | 1.1 | 0.8 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 |

# Curriculum Vitae

# Curriculum Vitae

## Personal Information

Full name: Jacob Strømberg
Date of birth: April 29, 1982
Place of birth: Værløse, Denmark

## Educational Background

| | |
|---|---|
| 2007 - 2013 | **University of Zürich (Department of Banking and Finance), Switzerland**<br>Swiss Finance Institute Ph.D. Program in Finance<br>Thesis: Essays on the Pricing and Modeling of Derivatives and Risk-taking Incentives |
| 2005 - 2007 | **University of Copenhagen (Department of Mathematical Sciences), Denmark**<br>Master of Science in Mathematics and Economics<br>Thesis: Modelling Yields and Exchange Rates - An Application to International Fixed Income Asset Allocation |
| 2001 - 2006 | **University of Copenhagen (Department of Mathematical Sciences), Denmark**<br>Bachelor of Science in Mathematics and Economics<br>Thesis: Calibration Methods for One-Factor Interest Rate Models - A Measure Change Approach |

## Professional Experience

| | |
|---|---|
| 2012 - | **UBS AG (Credit Methodology Lombard), Zürich, Switzerland**<br>Associate Director, Risk Modeling and Analytics Specialist |
| 2006 - 2007 | **European Central Bank (Risk Management Division), Frankfurt, Germany**<br>Economist and Internship in the Strategy Unit |
| 2004 - 2007 | **Danish National Bank (Statistics Department), Copenhagen, Denmark**<br>Part-time job as Junior Analyst in the Balance of Payments Division |
| 2003 - 2004 | **Danish Government (Ministry of the Interior and Health), Denmark**<br>Part-time job as Junior Analyst in an Economic Office for health |